RAINFOVER RAMSEY SIMPLE STRUCTURES

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Abstract. A relational structure $R$ is rainbow Ramsey if for every finite induced substructure $C$ of $R$ and every colouring of the copies of $C$ with countably many colours, such that each colour is used at most $k$ times for a fixed $k$, there exists a copy $R^*$ of $R$ so that the copies of $C$ in $R^*$ use each colour at most once.

We show that certain ultrahomogenous binary relational structures, for example the Rado graph, are rainbow Ramsey. Via compactness this then implies that for all finite graphs $B$ and $C$ and $k \in \omega$, there exists a graph $A$ so that for every colouring of the copies of $C$ in $A$ such that each colour is used at most $k$ times, there exists a copy $B^*$ of $B$ in $A$ so that the copies of $C$ in $B^*$ use each colour at most once.

1. Introduction

There is an extensive literature concerning colouring problems of the following type: Given conditions on the colouring function conclude that the restriction of the colouring function to a particular subset of its range is injective. In particular, this has been been widely studied for finite graphs, and then mostly complete graphs, and edge colourings. See [4] and [6] for two survey papers. Many of those problems and results are in analogy to standard Ramsey type problems and results, for which then the restriction of the colouring function to a particular subset of its range is asked to be constant. Those investigations have given rise to notions like anti-Ramsey numbers and restricted Ramsey numbers etc; and then results finding anti-Ramsey numbers for certain pairs of graphs, finding upper bounds, complexity, asymptotic behavior of the anti-Ramsey numbers. As is outlined in the two surveys cited above, problems in the context of rainbow Ramsey have been studied extensively for $k$-bounded colourings of pairs of natural numbers, from finding exact numbers, to finding growth rates, to investigations of the relative strength of the statement to other Ramsey properties; see [2] for relative strength investigation. There are a few articles extending the work to hypergraphs and a few to infinite graphs. See [7] and [5].

However, the literature has completely missed, for colouring of arbitrary finite substructures, finding rainbow copies of the Rado graph and other ultrahomogeneous structures, as well as investigating $k$-bounded colorings of copies of a fixed finite graph, rather than simply edge colorings. In this work, we address this void.

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For $B$ a relational structure we often denote by $B$ the set of its elements and sometimes use $B$ to denote the structure as well as the set of its elements. For $S \subseteq B$ let $B \downarrow S$ be the substructure of $B$ induced by $S$. For $B$ and $C$ relational structures denote by

$$\left( B \right)^C \left( C \right)$$

the set of induced substructures of $B$ isomorphic to $C$. The elements of $\left( B \right)^B$ are the copies of $B$ in $B$. An embedding of $B$ is an isomorphism of $B$ to a copy of $B$ in $B$.

Let $A$ and $C$ be relational structures and $k$ a natural number. A function $\gamma : \left( A \right)^C \rightarrow \omega$ is called $k$-bounded if $|\gamma^{-1}(n)| \leq k$ for every $n \in \omega$. A weaker notion and more appropriate for the arguments in this paper is that of a $k$-delta function. By a $\delta$-system of copies of $C$ in $A$ we mean a subset $S$ of $\left( A \right)^C$ for which $X \setminus \bigcap Y \in S$ is a singleton set for all $X \in S$. (This is a special case of the more general set-theoretic notion of $\delta$-system.) Then we call a function $\gamma : \left( A \right)^C \rightarrow \omega$ is $k$-delta if there is no $\delta$-system $S \subseteq \left( A \right)^C$ having $k + 1$ elements so that $\gamma$ is constant on $S$. Clearly every $k$-bounded function is $k$-delta.

The rainbow Ramsey properties we are interested in are defined as follows.

**Definition 1.1.** Let $A$, $B$ and $C$ be relational structures and $k$ a natural number. The arrow $A \xrightarrow{\text{rainbow}} \left( B \right)^C$ means that for every $k$-delta colouring $\gamma : \left( A \right)^C \rightarrow \omega$, there exists a $B^* \in \left( A \right)$ so that $\gamma$ is one-to-one on $\left( B^* \right)^C$.

The arrow $A \xrightarrow{\text{rainbow}} \left( B \right)^C$ means that for every $k$-bounded colouring $\gamma : \left( A \right)^C \rightarrow \omega$, there exists a $B^* \in \left( A \right)$ so that $\gamma$ is one-to-one on $\left( B^* \right)^C$.

Thus $A \xrightarrow{\text{rainbow}} \left( B \right)^C$ implies $A \xrightarrow{\text{rainbow}} \left( B \right)^C$.

A countably infinite relational structure $R$ is rainbow Ramsey if for every finite substructure $C$ and every $k \in \omega$ the relation $R \xrightarrow{\text{rainbow}} \left( R \right)^C$ holds; similarly $R$ is delta rainbow Ramsey if for every finite substructure $C$ and every $k \in \omega$ the relation $R \xrightarrow{\text{rainbow}} \left( R \right)^C$ holds. A class $\mathcal{F}$ of finite relational structures is rainbow Ramsey if for all structures $B$ and $C$ in $\mathcal{F}$ and every $k \in \omega$, there exists a structure $A \in \mathcal{F}$ for which relation (2) holds; similarly for delta rainbow Ramsey. If $R$ is a countably infinite (delta) rainbow Ramsey relational structure, then the class of finite structures isomorphic to an induced substructure of $R$, the age of $R$, $\text{Age}(R)$, is also (delta) rainbow Ramsey; see Corollary 5.1.

Relational structures $A$ and $B$ are called equimorphic, or said to be siblings, if there exists an isomorphic injection of $A$ into $B$ and an isomorphic injection of $B$ into $A$. Note that if $A$ and $B$ are siblings then for every $A^* \in \left( A \right)$ the set $\left( A^* \right)^B \neq \emptyset$ and for every $B^* \in \left( B \right)$ the set $\left( B^* \right)^A \neq \emptyset$. If $A$ and $B$ are equimorphic relational structures, then $A$ is (delta) rainbow Ramsey if and only if $B$ is (delta) rainbow Ramsey; see Lemma 5.2.
A binary relational structure is *simple* if for each of its symmetric relations the structure is a simple graph and for each of its directed relations the structure is the orientation of a simple graph; see Definition 3.1 for a more precise definition. The Rado graph is an example of a simple binary structure as is every simple graph. A structure is *ultrahomogeneous*, also often called simply *homogeneous*, if every isomorphism between finite substructures extends to an automorphism of the entire structure. See [3] for an introduction to homogeneous structures. We prove that simple ultrahomogeneous countable binary relational structures are (delta) rainbow Ramsey, as are other simple binary relational structures; see Theorem 5.2 for a more precise statement. It follows that the age of any (delta) rainbow Ramsey structure is also (delta) rainbow Ramsey, see Corollary 5.1. That such classes of simple finite binary relational structures are rainbow Ramsey also follows via a similar construction as given here from the Nešetřil-Rödl partition theorem; see [8]. In fact, any Fraïssé class of finite relational structures with free amalgamation and the Ramsey property is rainbow Ramsey, by a similar argument.

We conclude the introduction by pointing out that, while rainbow Ramsey holds for $k$-bounded colorings of the natural numbers as an easy consequence of Ramsey’s Theorem, and while for uncountable sets rainbow Ramsey results are independent of ZFC (see [10] and [1]), ours is the first work showing that homogeneous relational structures without the Ramsey property may still be rainbow Ramsey.

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2. Trees of Sequences

Let $\mathcal{T}_\omega$ be the tree consisting of all finite sequences with entries in $\omega$. The order of the tree $\mathcal{T}$, denoted by $\leq$, is given by sequence extension. That is $x = \langle x(0), x(1), \ldots, x(n-1) \rangle \leq y = \langle y(0), y(1), \ldots, y(m-1) \rangle$ if $n \leq m$ and $x(i) = y(i)$ for all $i \in n$. Let $S \subseteq \mathcal{T}_\omega$. The sequence $y \in S$ is an immediate $S$-successor of the sequence $x \in S$ if $x \subseteq y$ and there is no sequence $z \in S$ with $x \subset z \subset y$. The $S$-degree of $x \in S$ is the number of immediate $S$-successors of $x$. The length $|x|$ of the sequence $x = \langle x(0), x(1), \ldots, x(n-1) \rangle$ is $n$. The *intersection* $x \wedge y$ of two sequences is the longest sequence $z$ with $z \subseteq x$ and $z \subseteq y$. For $R \subseteq \mathcal{T}_\omega$ let $\operatorname{levels}(R) = \{|x| : x \in R \}$. For $|x| < |y|$ the number $y(|x|)$ is the *passing number* of $y$ at $x$.

We write $x \prec y$ if $x$ and $y$ are incomparable under $\subseteq$ and if

$$x(|x \wedge y|) < y(|x \wedge y|).$$

That is, for two incomparable sequences $x$ and $y$, if the passing number of $x$ is smaller than the passing number of $y$ at their intersection then $x \prec y$. For $\subseteq$ incomparable sequences the relation $\prec$ agrees with the lexicographic order on the tree $\mathcal{T}_\omega$.

As in the third paragraph of the preliminaries section of [9] and Definition 4.1 of [9], we define:

**Definition 2.1.** A subset $T$ of $\mathcal{T}_\omega$ is an $\omega$-tree if it is not empty, closed under initial segments, has no endpoints and the $T$-degree of every sequence $x \in T$ is finite.

For $2 \leq \kappa \in \omega$ let $\mathcal{T}_\kappa \subseteq \mathcal{T}_\omega$ be the tree consisting of all finite sequences with entries in $\kappa$. Then $\mathcal{T}_\kappa$ is an $\omega$-tree and actually a wide $\omega$-tree according to Definition 4.1 of [9]. The root of $\mathcal{T}_\kappa$ is the empty sequence $\emptyset$. 


Let $S \subseteq \mathfrak{T}$ be a set of sequences. Following Definitions 2.2 and 3.2 of [9] we define: The set $S$ is transversal if no two different elements of $S$ have the same length and it is an antichain if no two different elements of $S$ are ordered under the $\subseteq$ relation. The closure of $S$ is the set of intersections of elements of $S$. A subset $V \subseteq \mathfrak{T}$ is cofinal if for every $x \in \mathfrak{T}$ there is a $v \in V$ with $x \subseteq v$.

The set $S$ is diagonal if it is an antichain and the closure of $S$ is transversal and the degree of every element of closure $S$ within the closure of $S$ is at most 2. The set $S$ is strongly diagonal if it is diagonal and if for all $x, y, z \in S$ with $x \neq y$:

1. If $|x \land y| < |z|$ and $x \land y \not\subseteq z$ then $z(|x \land y|) = 0$.
2. $x(|x \land y|) \in \{0, 1\}$.

**Definition 2.2.** A function $f$ mapping a set $R \subseteq \mathfrak{T}$ of sequences into $\mathfrak{T}$ has the properties of Order, Level, Level-imp, Pnp (passing number preservation), Pnp-strong or Lexico respectively, if for all $x, y, z, u \in R$:

- **Order::** $x \land y \subseteq z \land u$ if and only if $f(x) \land f(y) \subseteq f(z) \land f(u)$.
- **Level::** $x \land y < |z \land u|$ if and only if $|f(x) \land f(y)| < |f(z) \land f(u)|$.
- **Level-imp::** $x \land y < |z \land u|$ implies $|f(x) \land f(y)| < |f(z) \land f(u)|$.
- **Pnp::** If $|z| > |x|$ then $z(|x|) = f(z)(|f(x)|)$.
- **Pnp-strong::** If $|z| > |x \land y|$ then $z(|x \land y|) = f(z)(|f(x) \land f(y)|)$.
- **Lexico::** If $x \prec y$ then $f(x) \prec f(y)$.

According to Definition 3.1 of [9], we define:

**Definition 2.3.** Let $R$ and $S$ be two subsets of $\mathfrak{T}_\omega$. A bijection $f$ of $R$ to $S$ is a strong similarity of $R$ to $S$ if it satisfies the properties of Order and Level andLexico which is implied by Pnp-strong; (and also Lexico which is implied by Pnp-strong).

The sets $R$ and $S$ are strongly similar, $R \approx S$, if there is a strong similarity, (it will be unique), of $R$ to $S$.

Then for $T$ a set of sequences, the set $\text{Sims}_T(R)$ is the equivalence class of all subsets $S$ of $T$ with $R \approx S$.

Observe that a strong similarity maps sequences of the same length to sequences of the same length. It can be viewed as “stretching” or in the inverse then “compressing” length of sequences, but preserving all shapes and passing numbers.

3. Binary structures encoded by $\mathfrak{T}$

**Definition 3.1.** A binary relational structure $R$ on a set $R$ with binary relations $E_0, E_1, \ldots, E_{n-1}$ and $n \geq 2$ is simple if:

1. Each of the binary relations $E_i$ is irreflexive.
2. There is a number $m \leq n$, the symmetry number of $R$ so that:
   a. $E_i(x, y)$ implies $E_i(y, x)$ for all $x, y \in R$ and all $i \in m$.
   b. $E_i(x, y)$ implies $\neg E_i(y, x)$ for all $x, y \in R$ and all $m \leq i \in n$.
3. For all $i, j \in n$ with $i \neq j$: $E_i(x, y)$ implies $\neg E_j(x, y)$ and $\neg E_j(y, x)$.
4. For all $(x, y)$, there is an $i \in n$ with $E_i(x, y)$.

That is, the set $R$ together with only one of the relations $E_i$ is a simple graph if $i \in m$, and is an orientation of a simple graph if $m \leq i \in n$. Any two different such graphs do not have overlapping edges. Note that every binary relational structure having only irreflexive relations can be encoded as a simple binary relational structure.
For $n$ and $m$ in $\omega$ and $m \leq n$ let $S(n,m)$ denote the class of all simple binary relational structures with set of binary relations $\{E_0,E_1,\ldots,E_n\}$ and symmetry number $m$. Let $A(n,m) \subseteq S(n,m)$ be the subclass of the finite structures in $S(n,m)$. The class $A(n,m)$ is an age with amalgamation whose Fräissé limit is denoted by $U(n,m)$. The ultrahomogeneous structures of the form $U(n,m)$ are the random simple binary ultrahomogeneous structures. Note that then $U(2,2)$ is the random graph. (It has adjacent and non-adjacent two element subsets as sets of relations.) Also, $U(1,0)$ is the random Tournament, $U(2,1)$ is the random oriented graph, $U(1,1)$ is the complete graph $K_\omega$, $U(0,0)$ is the relational structure having no relations, $U(3,3)$ may be seen as a graph having heavy and light edges and $U(3,2)$ as a graph having symmetric and oriented edges. The cases $(n,m) = (0,0)$ and $(n,m) = (1,1)$ do not quite fit into the general notational framework developed subsequently and will be dealt with later on. Note that for $m \leq n$ the inequality $(0,0) \neq (n,m) \neq (1,1)$ holds if and only if $m + 2(n - m) \geq 2$.

Let $m \leq n \in \omega$ and $\delta = m + 2(n - m) \geq 2$. The tree $T_\delta$ encodes a simple binary relational structure $T(n,m) \in S(n,m)$ on the set of sequences in $T_\delta$, in which for all $x,y \in T_\delta$ with $x \neq y$:

1. If $|x| < |y|$ then let for the passing number $p = y(|x|)$:
   (a) If $p \in m$ then $E_p(x, y)$ and $E_p(y, x)$.
   (b) If $p \geq m$ and $p - m$ even then $E_{m + \frac{p}{2}}(x, y)$.
   (c) If $p \geq m$ and $p - m$ odd then $E_{m + \frac{p-1}{2}}(y, x)$.

2. If $|x| = |y|$ and $x \prec y$ then if $m = 0$: $E_0(x,y)$. If $m > 0$: $E_0(x,y)$ and $E_0(y,x)$.

For $R$ a relational structure and $F$ a subset of $R$, the set of elements of $R$, let $R \downarrow F$ be the substructure of $R$ induced by $F$. Note that if $f$ is a similarity of a subset $F$ of $T_\delta$ to a subset $G$ of $T_\delta$ then $f$ is an isomorphism of $T(n,m) \downarrow F$ to $T(n,m) \downarrow G$.

In accordance with Definition 3.3 of [9]:

**Definition 3.2.** An injection $f : T_\delta \to T_\delta$ is a strong diagonalization of $T_\delta$ if the image of $f$ is a strongly diagonal subset of $T_\delta$ and if $f$ satisfies the properties of Level-imp and Pnp and Lexico.

According to Theorem 4.1 of [9] there exists a strong diagonalization of $T_\delta$ into $T_\delta$. We fix for every $2 \leq \delta \in \omega$ such a strong diagonalization $\Delta_\delta$ and denote by $D_\delta$ the image of the strong diagonalization $\Delta_\delta$. For $\delta = m + 2(n - m) \geq 2$ let $D(n,m) = T(n,m) \downarrow D_\delta$. Note that $\Delta_\delta$ is an isomorphism of $T(n,m)$ to $D(n,m)$. Hence $D(n,m)$ is equimorphic to $T(n,m)$; but is not a homogeneous structure.

**Theorem 3.1.** Let $m \leq n \in \omega$ and $m + 2(n - m) \geq 2$. The simple binary relational structures $U(n,m)$ and $T(n,m)$ and $D(n,m)$ are equimorphic.

**Proof.** Because $T(n,m)$ is equimorphic to $D(n,m)$ it suffices to show that $T(n,m)$ and $U(n,m)$ are equimorphic.

Every finite induced substructure of $T(n,m)$ is an element of $A(n,m)$. Hence and because the homogeneous structure $U(n,m)$ is universal, there exists an isomorphism of $T(n,m)$ into $U(n,m)$.

Let $\delta = m + 2(n - m)$ and $V$ be a transversal and cofinal subset of $T_\delta$. Let $F$ be a finite subset of $V$. Then for every function $f : F \to \delta$ there exists a sequence $y \in V$ so that the passing number $y(|x|) = f(x)$ for every $x \in F$. This then translates to the fact that the substructure of $T(n,m)$ induced by $V$ has the mapping extension.
property with respect to the age $A(n, m)$. Hence the structure induced by $V$ is homogeneous with age $A(n, m)$ and hence isomorphic to $U(n, m)$. \qed

4. Milliken

**Definition 4.1.** The set $T \subseteq \mathfrak{T}_\beta$ is closed by levels if $s \in T$ whenever there exist $t \in T$ with $s \subseteq t$ and $y \in T$ with $|s| = |y|$. Let $T$ be a meet closed set of sequences which is also closed by levels and let $n \in \omega + 1$. The set $S \subseteq T$ is an element of $\text{Str}^n(T)$ if:

1. $|\text{levels}(S)| = n$.
2. $S$ is meet closed and closed by levels.
3. For all $s \in S$, the degree of $s$ in $S$ is 0 or equal to the degree of $s$ in $T$.

Note the following:

- The tree $\mathfrak{T}_\beta$ is meet closed and closed by levels. Let $T \subseteq \mathfrak{T}_\beta$ be meet closed, closed by levels, non-empty and having no endpoints. Then for $S \in \text{Str}^\omega(T)$ removing, from every sequence $s \in S$, the entries whose indices are not in levels$(S)$ results in the tree $\mathfrak{T}_\beta$. For every $S \in \text{Str}^\omega(T)$ there exists a unique strong similarity $f : T \to S$. This similarity $f$ maps every level of $T$ into a level of $S$, preserving the order, via length, of the levels. Conversely, for every strong similarity $f$ of $T$ into $T$, the set $f[T] \in \text{Str}^\omega(T)$. If $R \in \text{Str}^\omega(S)$ and $S \in \text{Str}^\omega(T)$ then $R \in \text{Str}^\omega(T)$. For $T \in \text{Str}^\omega(\mathfrak{T}_\beta)$ let $T^* := \{x \in \mathfrak{T}_\beta \mid \exists y \in T (x \subseteq y)\}$. Then $T^*$ is an $\omega$-tree and if $T^* = S^*$ then $T = S$.

Hence the following Lemma 4.1 is a direct consequence of Theorem 5.2 of [9].

**Lemma 4.1.** Let $T = f[\mathfrak{T}_\beta]$ for $f$ a strong similarity of $\mathfrak{T}_\beta$ to $T \subseteq \mathfrak{T}_\beta$. Let $F$ be a finite meet closed subset of $T$ and $m \in \omega$. Then for any colouring $\gamma : \text{Sims}_T(F) \to m \in \omega$ there exists a strong similarity function $f$ of $T$ to a tree $S \subseteq T$ so that $\gamma$ is constant on $\text{Sims}_S(F)$.

The following thus follows from the fact that if $F$ and $G$ are two finite and strongly diagonal sets with $F \not\sim G$ and with closure$(F) = \text{closure}(G)$ then $F = G$.

**Corollary 4.1.** Let $T = f[\mathfrak{T}_\beta]$ for $f$ a strong similarity of $\mathfrak{T}_\beta$ to $T \subseteq \mathfrak{T}_\beta$. Let $F$ be a finite and strongly diagonal subset of $T$ and $m \in \omega$. Then for any colouring $\gamma : \text{Sims}_T(F) \to m \in \omega$ there exists a strong similarity function $f$ of $T$ to a tree $S \subseteq T$ so that $\gamma$ is constant on $\text{Sims}_S(F)$.

By repeated application of the above Lemma 4.1 and Corollary 4.1 we obtain:

**Theorem 4.1.** Let $\{F_i \mid i \in p \in \omega\}$ be a finite set of finite meet closed subsets of $\mathfrak{T}_\beta$ so that $\neg(F_i \not\sim F_j)$ for $i \neq j$. Let $m_i \in \omega$ for all $i \in p$. Then for any set $\gamma_i : \text{Sims}_\mathfrak{T}_\beta(F_i) \to m_i$ of colouring functions there exists a strong similarity function $f$ of $\mathfrak{T}_\beta$ to a tree $T \subseteq \mathfrak{T}_\beta$ so that each one of the colouring functions $\gamma_i$ is constant on $\text{Sims}_T(F_i)$.

**Corollary 4.2.** Let $\{F_i \mid i \in p \in \omega\}$ be a finite set of strongly diagonal subsets of $\mathfrak{T}_\beta$ so that $\neg(F_i \not\sim F_j)$ for $i \neq j$. Let $m_i \in \omega$ for all $i \in p$. Then for any set $\gamma_i : \text{Sims}_\mathfrak{T}_\beta(F_i) \to m_i$ of colouring functions there exists a strong similarity function $f$ of $\mathfrak{T}_\beta$ to a tree $T \subseteq \mathfrak{T}_\beta$ so that each one of the colouring functions $\gamma_i$ is constant on $\text{Sims}_T(F_i)$.

Hence we can apply these results to $\mathfrak{D}_\beta$. 

\[ \text{□} \]
Theorem 4.2. Let \( \{F_i \mid i \in p \in \omega \} \) be a finite set of finite subsets of \( \mathcal{D}_\omega \) so that \( \neg(F_i \sim F_j) \) for \( i \neq j \). Let \( m_i \in \omega \) for all \( i \in p \). Then for any set \( \gamma_i : \text{Sims}_{\mathcal{D}_\omega}(F_i) \to m_i \) of colouring functions there exists a strong similarity function \( g \) of \( \mathcal{D}_\omega \) to a tree \( D \subseteq \mathcal{D}_\omega \) so that each of the colouring functions \( \gamma_i \) is constant on \( \text{Sims}_D(F_i) \).

Proof. For every \( i \in p \) let \( \delta_i : \text{Sims}_{\mathcal{T}_\omega}(F_i) \to m_i \) be given by \( \delta_i(F'_i) = \gamma_i(\Delta_\omega(F'_i)) \) for \( F'_i \in \text{Sims}_{\mathcal{T}_\omega}(F_i) \). Observing that \( F_i \sim F'_i \) if and only if \( F_i \sim \Delta_\omega(F'_i) \) because \( F_i \) and hence \( F'_i \) are strongly diagonal.

According to Corollary 4.2 there exists a strong similarity function \( f \) of \( \mathcal{T}_\omega \) to a tree \( T \subseteq \mathcal{T}_\omega \) so that each one of the colouring functions \( \delta_i \) is constant on \( \text{Sims}_T(F_i) \). The restriction of \( \Delta_\omega \) to a strongly diagonal subset \( S \) of \( \mathcal{T}_\omega \) is a strong similarity of \( S \). Hence the function \( g \), which is the restriction of \( \Delta_\omega \circ f \) to \( \mathcal{D}_\omega \) is a strong similarity of \( \mathcal{D}_\omega \) to a subset \( D \) of \( \mathcal{D}_\omega \). \( \square \)

Let \( n \geq 2 \) and \( d = m + 2(n - m) \). Let \( R \) and \( S \) be two finite induced substructures of \( \mathcal{D}(n, m) \) with domains \( R, S \subseteq \mathcal{D}_\omega \), the set of elements of \( R \) and \( S \) respectively. That is \( R = \mathcal{D}(n, m) \upharpoonright R \) and \( S = \mathcal{D}(n, m) \upharpoonright S \). Then \( R \) and \( S \) are strongly similar (as substructures), written again \( R \sim S \), if their domains are strongly similar, that is \( R \sim S \) then the strong similarity of \( R \) to \( S \) is an isomorphism of the binary structure \( R \) to the binary structure \( S \). For \( S \) an induced substructure of \( \mathcal{D}(n, m) \) and \( F \) a finite induced substructure of \( S \) let \( \text{Sims}_S(F) \) be the set of all induced substructures \( F' \) of \( S \) with \( F \sim F' \). Then \( \text{Sims}_S(F) \) is the set of all \( F' \in \binom{S}{F} \) with \( F \sim F' \). Hence:

Corollary 4.3. Let \( n \geq 2 \) and \( d = m + 2(n - m) \). Let \( \{F_i \mid i \in p \in \omega \} \) be a finite set of finite induced substructures of \( \mathcal{D}(n, m) \) so that \( \neg(F_i \sim F_j) \) for \( i \neq j \). Let \( m_i \in \omega \) for all \( i \in p \).

Then for any set \( \gamma_i : \text{Sims}_{\mathcal{D}(n, m)}(F_i) \to m_i \) of colouring functions there exists a strong similarity function \( g \) of \( \mathcal{D}(n, m) \) to a copy \( D \) of \( \mathcal{D}(n, m) \) so that each one of the colouring functions \( \gamma_i \) is constant on \( \text{Sims}_D(F_i) \).

5. \( \mathcal{T}(n, m) \) and \( \mathcal{U}(n, m) \) and \( \mathcal{D}(n, m) \) are rainbow Ramsey.

In this section we show that simple relational structures are \( k \)-delta rainbow Ramsey for any \( k \), and thus rainbow Ramsey.

First we show that it is sufficient to show the result for \( k = 2 \).

Lemma 5.1. Let \( R \) be a relational structure and let \( C \in \text{Age}(R) \).

If \( R \xrightarrow{\text{rainbow}} \langle R \rangle^C \), then \( R \xrightarrow{\text{rainbow}} \langle R \rangle^C \) for all \( k \geq 2 \).

Proof. Let \( C \in \text{Age}(R) \). The proof is by induction on \( k \geq 2 \), where the base case is given by the hypothesis.

Thus assume that \( R \xrightarrow{\text{rainbow}} \langle R \rangle^C \). Let \( f : \binom{R}{C} \to \omega \) be a \( k + 1 \)-delta colouring. Note that if \( S \) is a \( \delta \)-system of copies of \( C \), then \( | \bigcap_{Y \in S} Y | = |C| - 1 \). Thus, each maximal \( \delta \)-system \( S \) is uniquely determined by the structure \( \bigcap_{Y \in S} Y \); for \( S \) is the collection of all copies \( X \) of \( C \) in \( \text{Age}(R) \) for which \( \bigcap_{Y \in S} Y \) is an induced substructure of \( X \).

Enumerate the members of \( \binom{R}{C} \) as \( C_n, n \in \omega \), and enumerate the maximal \( \delta \)-systems of copies of \( C \) as \( S_i, i \in \omega \). Let \( \ell \in \omega \) be fixed. If there are \( k + 1 \) members
of \( S_0 \) with \( f \)-colour \( \ell \), then choose \( n(0, \ell) \) to be the least \( n \) such that \( C_n \in S_0 \) and \( f(C_n) = \ell \), and let \( N_{0, \ell} = \{ n(0, \ell) \} \). Otherwise, let \( N_{0, \ell} = \emptyset \).

Suppose we have chosen \( N_{i, \ell} \). If either (a) there are \( k + 1 \) members of \( S_{i+1} \) with \( f \)-colour \( \ell \) and there is an \( n \in N_{i, \ell} \) such that \( C_n \in S_{i+1} \) and \( f(C_n) = \ell \), or (b) there are at most \( k \) members of \( S_{i+1} \) with \( f \)-colour \( \ell \), then let \( N_{i+1, \ell} = N_{i, \ell} \). Otherwise, there are \( k + 1 \) members of \( S_{i+1} \) with \( f \)-colour \( \ell \) and none of these members have an index in \( N_{i, \ell} \). Then let \( n(i+1, \ell) \) be the least \( n \) such that \( C_n \in S_{i+1} \) with \( f(C_n) = \ell \), and set \( N_{i+1, \ell} = N_{i, \ell} \cup \{ n(i+1, \ell) \} \).

At the end of the inductive construction, let \( N_\ell = \bigcup_{i \in \omega} N_{i, \ell} \). Note that this construction ensures that for each \( i \in \omega \) for which \( S_i \) has \( k + 1 \) members with \( f \)-colour \( \ell \), there is exactly one \( n \in N_i \) such that \( C_n \in S_i \). Note further that for \( \ell \neq \ell' \), \( N_\ell \cap N_{\ell'} = \emptyset \).

Let \( N = \bigcup_{i \in \omega} N_i \) and define a \( k \)-delta colouring \( g \) on \( \binom{R}{C} \) as follows. For each \( n \in \omega \setminus N \), let \( g(C_n) = f(C_n) \); and for \( \ell \in N \), let \( g(C_n) = \omega + n \). Then \( g \) is \( k \)-delta, since if \( S_i \) has \( k + 1 \) members with \( f \)-colour \( \ell \), then there is exactly one \( n \in N_i \) such that \( C_n \in S_i \), and \( g(C_n) \) is defined to be \( \omega + n \). By the induction hypothesis, there is a copy \( R^* \in \binom{R}{C} \) such that \( g \) is one-to-one on \( \binom{R}{C} \). Then \( f \) is \( 2 \)-delta on \( \binom{R}{C} \). By the induction hypothesis applied again, there is another copy \( R^* \) in \( \binom{R}{C} \) such that \( f \) is one-to-one on \( \binom{R}{C} \).

**Proposition 5.1.** Let \( m \leq n \in \omega \) and \( m + 2(n - m) \geq 2 \), and let \( C \) be a finite induced substructure of \( D(n, m) \). Then

\[
D(n, m) \xrightarrow{\text{rainbow}} \binom{D(n, m)}{C}.
\]

**Proof.** Let \( \mathcal{D} = m + 2(n - m) \) and \( C \subseteq \mathcal{D} \) with \( C = D(n, m) \downarrow C \). Let \( B \) be the set of triples \( (B, X, Y) \) of subsets of \( \mathcal{D} \) for which \( B = X \cap Y \) and \( X \neq Y \) and \( X \preceq C \preceq Y \). To easily identify the order of the copies we require further that the longest sequence \( x \in X \cap Y \) is an element of \( X \). Write \( (B, X, Y) \preceq (B', X', Y') \) if \( B \preceq B' \), and moreover write \( (B, X, Y) \equiv (B', X', Y') \) if \( B \preceq B' \) and if \( f[X] = X' \) and \( f[Y] = Y' \) for the (unique) strong similarity \( f \) of \( B \) to \( B' \). Note that if \( g \) is a strong similarity function of \( \mathcal{D} \) onto a subset of \( \mathcal{D} \) and if \( (B, X, Y) \equiv (B', X', Y') \) then \( (B, X, Y) \equiv (g(B), g(X), g(Y)) \equiv (g(B'), g(X'), g(Y')) \). Also, if \( (B, X, Y) \equiv (B, X', Y') \) then \( X = X' \) and \( Y = Y' \).

Enumerate the finitely many \( \equiv \) equivalence classes as \( \mathcal{P}_i \) for \( i \in \omega \). Let \( B_0 \) denote the set of all \( B \) for which there exist \( X \) and \( Y \) such that \( (B, X, Y) \in \mathcal{B} \). Observe that for every \( B \in \mathcal{B}_0 \) and every \( i \in \omega \), either there is no pair \( (X, Y) \) of copies of \( C \) with \( (B, X, Y) \in \mathcal{P}_i \), or else there exists exactly one such pair.

Let \( i \in \omega \) and \( (B, X, Y) \in \mathcal{P}_i \) be given. We claim that there are at least three distinct \( (B_j, X_j, Y_j), j \leq 2 \), such that:

1. each \( (B_j, X_j, Y_j) \equiv (B, X, Y) \), and hence are in \( \mathcal{P}_i \);
2. \( X_0, X_1, X_2 \) are distinct and form a \( \delta \)-system;
3. \( Y_0 = Y_1 = Y_2 \).

The three distinct sets will be obtained by stretching some appropriate elements. First since \( B \) is a subset of \( \mathcal{D} \), \( B \) is also a subset of \( \mathcal{D} \) and is strongly diagonal. Let \( x \) denote the longest element in \( B \) with \( x \notin X \cap Y \); without loss of generality.
we may assume that \( x \in X \). Now for \( z \in B \), define \( \Lambda_z(z) = \)

\[
\begin{cases}
  z & \text{if } |z| < |x|; \\
  (z(0), \ldots, z(|x| - 1), z(|x|), z(|x| + 1), \ldots, z(|z| - 1)) & \text{if } |z| > |x|.
\end{cases}
\]

Now define \( x^0 = x \cdot 0 \), \( x^1 = x \cdot 1 \), and \( x^2 = x \). For each \( j \leq 2 \), let

\[
\begin{align*}
  B^j &= \{ \Lambda_z(z) : z \in B \setminus \{ x \} \} \cup \{ x^j \} \\
  X^j &= \Lambda_z(X \setminus \{ x \}) \cup \{ x^j \} \\
  Y^j &= \Lambda_z(Y).
\end{align*}
\]

Then each \( B^j \) is strongly diagonal, \( B^j \sim B \), and moreover, \((B^j, X^j, Y^j) \equiv (B, X, Y)\). Let \( B_j, X_j, Y_j \) denote \( \Delta_\delta(B^j), \Delta_\delta(X^j), \Delta_\delta(Y^j) \), respectively. Since \( \Delta_\delta \) is a strong diagonalization of \( \mathcal{D}_\delta \), each \((B_j, X_j, Y_j), j \leq 2 \) is in \( \mathcal{P}_i \). The sets \( X_0, X_1, X_2 \) form a \( \delta \)-system of copies of \( C \). It follows that for any strong similarity \( f, f(\mathcal{P}_i) \) contains three different \( \equiv \) related elements with the same third entry whose three second entries form a \( \delta \)-system of copies of \( C \).

Let \( \delta : \text{Sims}_{B(n, m)}(C) \to \omega \) be a 2-delta colouring. Let \( \lambda : \text{Sims}_{\mathcal{D}_\delta}(C) \to \omega \) be given by \( \lambda(C) = \delta(C) \) for \( C = B(n, m) \downarrow \mathcal{D}_\delta \). Then \( \lambda \) is a 2-delta colouring of \( \text{Sims}_{\mathcal{D}_\delta}(C) \). Associate with every \( B \in \mathcal{B}_0 \) the \( r \)-tuple \( \gamma(B) = (\sigma_i(B) ; i \in r) \), with entries one of the elements in the set \( \{ =, \neq, \not\in \} \), so that for \( i \in r \):

\[
\sigma_i(B) = \begin{cases}
  = & \text{ if } (B, X, Y) \in \mathcal{P}_i \text{ for sets } X \text{ and } Y \text{ with } \lambda(X) = \lambda(Y) , \\
  \neq & \text{ if } (B, X, Y) \in \mathcal{P}_i \text{ for sets } X \text{ and } Y \text{ with } \lambda(X) \neq \lambda(Y) , \\
  \not\in & \text{ if there are no sets } X \text{ and } Y \text{ with } (B, X, Y) \in \mathcal{P}_i .
\end{cases}
\]

According to Theorem 4.2 there exists a strong similarity function \( g \) of \( \mathcal{D}_\delta \) to a tree \( D \subseteq \mathcal{D}_\delta \) so that the colouring function \( \gamma \) is constant on every \( \sim \)-equivalence class of subsets in \( \mathcal{B}_0 \). Let \( \mathcal{M} \) denote the set of all the copies of \( B \) in \( D \).

If there is an \( i \in r \) and a \( B \in \mathcal{M} \) for which \( \sigma_i(B) \) is equal to \( = \), then there are sets \( X \) and \( Y \) with \((B, X, Y) \in \mathcal{P}_i \) and with \( \sigma_i(X) = \sigma_i(Y) \). By the above the \( \equiv \)-equivalence class containing \((B, X, Y) \) contains (at least) three elements \((B_j, X_j, Y_j) \), \( j \leq 2 \), with the \( X_j, j \leq 2 \), distinct and moreover forming a \( \delta \)-system of copies of \( C \). Then \((g[B_j], \{g[X_j] \}, g[Y_j]) \), \( j \leq 2 \), are three triples for which \( g[X_0], g[X_1], g[X_2] \) form a \( \delta \)-system of copies of \( C \) with \( \lambda(g[X_0]) = \lambda(g[X_1]) = \lambda(g[X_2]) \), contradicting that \( \lambda \) is 2-delta.

Thus \( \sigma_i(B) \) is not equal to \( = \) for every \( B \in \mathcal{M} \) and every \( i \in r \). Now suppose \( X \) and \( Y \) are copies of \( C \) in \( D \), the \( g \)-image of \( \mathcal{D}_\delta \). Letting \( B = X \cup Y \), the triple \((B, X, Y) \) is in \( \mathcal{P}_i \) for some \( i \in r \), and thus \( \lambda(X) \neq \lambda(Y) \).

This completes the proof.

We now observe that (delta) rainbow Ramsey is preserved among equimorphic structures.

**Lemma 5.2.** Let \( A \) and \( B \) be two equimorphic structures. Then

\[
A \xrightarrow{k\text{-delta}} (A)^C \quad \text{if and only if} \quad B \xrightarrow{k\text{-delta}} (B)^C .
\]

Similarly for the rainbow Ramsey property.

**Proof.** By symmetry it suffices to assume that \( A \) is an induced substructure of \( B \).
Let $A \xrightarrow{k\text{-delta}} (A)^C$ and $\gamma : (B)^C \to \omega$ be $k$-delta. The restriction of $\gamma$ to $(A)^C$ is $k$-delta and hence there a copy $A^* \in (A)^C$ so that $\gamma$ is one to one on $(A)^C$. Let $B^* \in (B)^C$. Then $\gamma$ is one to one on $(B)^C$.

We now come to the main result of this paper.

**Theorem 5.2.** The binary ultrahomogeneous structure $U(n, m)$ is (delta) rainbow Ramsey for all $m$ and $n$ in $\omega$ with $m \leq n$.

If $m \leq n \in \omega$ and $m + 2(n - m) \geq 2$, then the tree structure $T(n, m)$ and its diagonal substructure $D(n, m)$ of $T(n, m)$ are (delta) rainbow Ramsey as well.

**Proof.** If $m \leq n \in \omega$ and $m + 2(n - m) \geq 2$, the Theorem follows from Proposition 5.1 in conjunction with Lemma 5.1 and then Theorem 3.1 in conjunction with Lemma 5.2.

The special cases $U(0, 0)$ and $U(1, 1)$ follow from the standard Ramsey theorem in a similar but much simpler way as in the proof of Proposition 5.1; see [2].

By a standard compactness, the general (delta) rainbow Ramsey result implies a finite version.

**Lemma 5.3.** Let $R$ be a countably infinite relational structure for which the relation $R \xrightarrow{k\text{-delta}} (R)^C$ holds. Then for each $B \in \text{Age}(R)$ there is an $A \in \text{Age}(R)$ such that the relation $A \xrightarrow{k\text{-delta}} (B)^C$ holds.

The corresponding result holds for rainbow Ramsey.

**Proof.** Let $(r_i : i \in \omega)$ be an $\omega$-enumeration of $R$ and for $n \in \omega$ let $R_n$ be the induced substructure of $R$ on the set $\{r_i : i \in n\}$. Let $B \in \text{Age}(R)$ and assume for a contradiction that for every $n \in \omega$ there is a $k$-delta colouring $\gamma_n : (R)^C_n \to \omega$ for which the restriction of $\gamma_n$ to $(B)^C$ is not one-to-one for every $B^* \in (R_B^*)$. We will construct a $k$-delta colouring $\gamma : (R)^C \to \omega$ so that for all $B^* \in (R_B^*)$ the colouring $\gamma$ is not injective on $(B)^C$; contradicting $R \xrightarrow{k\text{-delta}} (R)^C$.

Let $\mathcal{U}$ be an ultrafilter on $\omega$ which contains all co-finite subsets of $\omega$. For $C^* \in (R)^C$ and $C^\circ \in (R)^C$ let $C^*$ and $C^\circ$ be equivalent, $C^* \sim C^\circ$, if $\{n \in \omega : \gamma_n(C^*) = \gamma_n(C^\circ)\} \in \mathcal{U}$. Let $\gamma : (R)^C \to \omega$ be a function which is constant on every $\sim$-equivalence class and assigns different colours to elements in different $\sim$-equivalence classes.

We claim that the function $\gamma$ is delta. For assume that $S$ is a $\delta$-system of structures containing $k$ structures in $(R)^C$ and that $\gamma$ is constant on $S$. This is not possible because there is then an $n \in \omega$ so that $\gamma_n$ is constant on $S$.

Finally we must show that there is no $B^* \in (R_B^*)$ for which $\gamma$ is injective on $(B)^C$.

To do so assume otherwise there is such a $B^*$; but because $(B)^C$ is finite, there is an $n \in \omega$ for which $\gamma_n$ is injective on $B^*$, a contradiction.

**Corollary 5.1.** If $R$ is a countably infinite (delta) rainbow Ramsey relational structure, then the class of finite structures isomorphic to an induced substructure of $R$ is also (delta) rainbow Ramsey.
6. Conclusion

In this paper we proved rainbow Ramsey results for simple binary relational structures, leaving open a wide spectrum of relational structures. Thus we can easily ask the following.

Questions 6.1. (1) Which homogeneous relational structures have the rainbow Ramsey property?
(2) Are there some ages which have the rainbow Ramsey property, but the Fraïssé limit does not? Or are the two equivalent?

In particular we do not know whether the homogeneous triangle-free graph has the rainbow Ramsey property, and we can ask for which finite triangle-free graph \(C\), does the countable triangle-free homogeneous graph have the rainbow Ramsey property.

In [2], Csima and Mileti proved that the reverse mathematical strength of the Rainbow Ramsey Theorem for pairs of natural numbers is strictly weaker over \(\text{RCA}_0\) than Ramsey’s Theorem. We may ask similar questions for the Rado graph and other homogeneous relational structures, replacing Ramsey’s Theorem with the existence of canonical partitions in the sense of Corollary 4.3.

Questions 6.2. (1) For homogeneous binary simple structures, is the statement of Corollary 4.3 stronger over \(\text{RCA}_0\) than the statement of Theorem 5.2?
(2) In particular, what is the reverse mathematical strength of the Rado graph being rainbow Ramsey for \(k\)-bounded colorings of edges; how does its strength compare with those of Ramsey’s Theorem and the Rainbow Ramsey Theorem for \(k\)-bounded colorings of pairs of natural numbers?

References

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