Isomorphisms of Graphs

**Definition:** If $A$ and $B$ are sets, then a **one-to-one correspondence** between $A$ and $B$ is an association of elements of $A$ with elements of $B$ in such a way the members of $A$ can be "evenly matched" with the members of $B$. Evenly matched means that each member of $A$ is paired with one and only one member of $B$, each member of $B$ is paired with one and only one member of $A$, and none of the members from either set are left unpaired. The result is that every member of $A$ is paired with exactly one member of $B$, and every member of $B$ is paired with exactly one member of $A$.

**Example:** If $A = \{1, 2, 3, 4, 5\}$ and $B = \{v, w, x, y, z\}$ then a one-to-one correspondence between them could be

1 $\leftrightarrow v$
2 $\leftrightarrow x$
3 $\leftrightarrow w$
4 $\leftrightarrow z$
5 $\leftrightarrow y$

another one would be

1 $\leftrightarrow z$
2 $\leftrightarrow y$
3 $\leftrightarrow x$
4 $\leftrightarrow w$
5 $\leftrightarrow v$

and we could come up with many others.

**Definition.** Two graphs $G$ and $H$ are said to be **isomorphic** if:
1.) there is a one-to-one correspondence between their vertex sets, and
2.) whenever two vertices are adjacent in $G$, the corresponding two vertices are adjacent in $H$.

We call the correspondence between $G$ and $H$ an isomorphism, and we write $G \cong H$ to denote that $G$ and $H$ are isomorphic.

Isomorphisms preserve several different aspects of a graph. From the definition, isomorphisms preserve vertex adjacency; that is, if an isomorphism corresponds vertices $A$ and $B$ in one graph with $X$ and $Y$ in another graph, then if there is an edge $\{A, B\}$ in the first graph, there will be an edge $\{X, Y\}$ in the second graph, and if there isn't an edge connecting $A$ to $B$ in the first graph, there won't be one connecting $X$ and $Y$, either.

Examples of isomorphic graphs:
is isomorphic to correspondence: $P \leftrightarrow 1, Q \leftrightarrow 2, R \leftrightarrow 3, S \leftrightarrow 6, T \leftrightarrow 5, U \leftrightarrow 4$. Verify that the adjacencies are all preserved as well.

Another way to think of isomorphism is as follows: $G$ and $H$ are isomorphic if either
1.) they are equal, or
2.) they could be made equal by changing the labels of the vertex set for one of them.

Example:

An isomorphism here could be: $A \leftrightarrow 2, B \leftrightarrow 3, C \leftrightarrow 4, D \leftrightarrow 1$.

Example:
Another way to see isomorphism: If you can re-draw one of the graph diagrams so the two have identical diagrams which only differ by names of the vertices. We could also say that if you can "deform" the one graph into the shape of the other, that is a sign of an isomorphism.

Example:

![Graph Diagram 1](image1)

Properties preserved by isomorphism:
1.) Number of vertices
2.) Number of edges
3.) Distribution of degrees
4.) The number of "pieces" of a graph

If a pair of graphs $G$ and $H$ differ in one or more of these characteristics, then $G$ and $H$ are NOT isomorphic.