Music from Mathematics

Music often has Melody and accompanying Chords

Pleasant musical Tones are called consonant

This Talk is about Theories of Consonance of musical tones proposed by Rameau, Helmholtz and the Greek Philosophers called `Pythagoreans'

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The Pythagoreans

- Ratios of small integers
- Scales of frequency rations of small integers
- Octave 2:1
- Quintic 3:2
- Quartic 4:3
- Can construct whole scale of 12 tones
- Pythagorean Comma: $2^7$ is not equal to $(\frac{3}{2})^{12}$
Jean-Philippe Rameau

- *Treatise on Harmony*, 1722
- Suggested that chord should have easily determinable fundamental bass
- For example: c1 e1 g1 are frequency multiples of C1
- So C is the `Basse fundamentale' for this chord
- Ernst Terhardt: “Subharmonic Coincidence”
  - J.F. Shouten: *The Perception of Subjective Tones* KNAW 41, 1051, 1974
Consonance due to (lack of) disturbing effect of beats

Two pure (sinusoidal) tones of similar frequency display beats.

If beats become themselves an audible tone, they become audible in the form of `roughness' within a `critical frequency range'

M.Mathews, J.Pierce: *Harmony and nonharmonoic partials*, 1980, Some evidence for both theories
no upper partial or combinational tones intervene. According to the assumptions made in the last chapter respecting the degree of damping possessed by Corti's organs (p. 144c), it would result, for example, that for the interval of a whole Tone \( c \, d \), such of Corti's fibres as have the proper tone \( c^\# \), would be excited by each of the tones with \( \frac{1}{10} \) of its own intensity; and these fibres will therefore fluctuate between the intensities of vibration 0 and \( \frac{1}{10} \). But if we strike the simple tones \( c \) and \( c^\# \), it follows from the table there given that Corti's fibres which correspond to the middle between \( c \) and \( c^\# \) will alternate between the intensities 0 and \( \frac{1}{10} \). Conversely the same intensity of beats would for a minor Third amount to only 0.194, and for a major Third to only 0.108, and hence would be scarcely perceptible beside the two primary tones of the intensity 1.

Fig. 59, which we used on p. 144d to express the intensity of the sympathetic vibration of Corti's fibres for an increasing interval of tone, may here serve to show the intensity of the beats which two tones excite in the ear when forming different intervals in the scale. But the parts on the base line must now be considered to represent fifths of a whole Tone, and not as before of a Semitone. In the present case the distance of the two tones from each other is doubly as great as that between either of them and the intermediate Corti's fibres.

Had the damping of Corti's organs been equally great at all parts of the scale, and had the number of beats no influence on the roughness of the sensation, equal intervals in all parts of the scale would have given equal roughness to the combined
Beats and Roughness of pure Tones

Helmholtz' Results:

Adding up pairs of pure partials give more or less rough intervals of pairs of complex tones

From H. Helmholtz: *The sensations of Musical Tones*, 1887
The experiment of Mathews and Pierce

- *Scaled* complex tones sound worse than original ones. Supporting:

  Rameau is right not Helmholtz
The experiment of Mathews and Pierce

- However: Conditioning ('Brainwashing') may be cause
- Experiment also shows high finality rating for a cadence of two chords of which first all the partials produce roughness and then none do:

If high partials are given larger weight, then 'finality' correlates more with
Reduction of roughness in final cadences of classical pieces of western music.
Helmholtz' calculation of Roughness

<table>
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<tr>
<th>Intervals</th>
<th>No.</th>
<th>Helmholtz's Notation, as in Diagram</th>
<th>Ellia's Notation of Intervals reckoned from c</th>
<th>Ratio</th>
<th>Cents</th>
<th>Roughness</th>
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<td></td>
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Helmholtz' calculation of Roughness

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Sound Examples


- Beats:
  Amplitude is oscillating at difference-frequency
- Beats become unpleasant only if in critical bandwidth

\[ \sin(a) + \sin(a+h) = 2 \sin(a + h/2) \cos(h) \]

- How can one define musical distance between tones?
Musical Consonance \textit{and} Distance

Let chord be represented by vector $c$

Let consonance be \textquoteleft length\textquoteright $\|c\|$

$\|c\|^2 = (c, Kc)$

$K_{ij} =$ observed consonance due to (lack of) roughness of all involved partials of instrument

Scalar Product must be positive!

We observe for metric given by

$\|c - d\|$

that consonant triads which appear together in classical music are \textquoteleft close\textquoteright !
Landscapes of Roughness

From F. Sobieczky: *Visualization of Roughness in Musical Consonance*, 1996

**Figure 7**: Tracing chords in a consonance diagram

C stands for consonance, and $c_1, c_2$ are the number of semitones from the base tone. The first three chords have the coordinates $(c_1, c_2) = (4, 7), (4, 12), (4, 16)$ at which the consonance clearly shows extremal values. The second progression of three chords belongs to $(c_1, c_2) = (2, 9), (2, 14), (2, 7)$, and the third to $(c_1, c_2) = (3, 7), (3, 14), (3, 17)$. 
Steepest descent of roughness
Computer Program to calculate `Forces' on keys of triads towards less roughness. C[] is the dominant-sept chord c g a#. The forces from result `point' towards c a c1.