The universal triangle-free graph has finite big Ramsey degrees

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Ramsey's Theorem

Ramsey's Theorem. Given any $k, l \geq 1$ and a coloring on the collection of all k-element subsets of $\mathbb{N}$ into l colors, there is an infinite set M of natural numbers such that each k-element subset of M has the same color.

Ramsey's Theorem and its finite version has been extended to many types of structures.

Structural Ramsey Theory looks for a large substructures of certain forms inside a given structure on which colorings are simple.

We'll now look at structural Ramsey theory on graphs.

Graphs and Ordered Graphs

Graphs are sets of vertices with edges between some of the pairs of vertices.

An ordered graph is a graph whose vertices are linearly ordered.

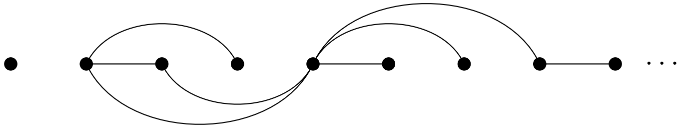


Figure: An ordered graph B

Embeddings of Graphs

An ordered graph A embeds into an ordered graph B if there is a one-to-one mapping of the vertices of A into some of the vertices of B such that each edge in A gets mapped to an edge in B, and each non-edge in A gets mapped to a non-edge in B.

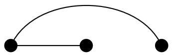


Figure: A

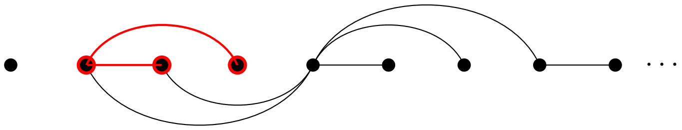
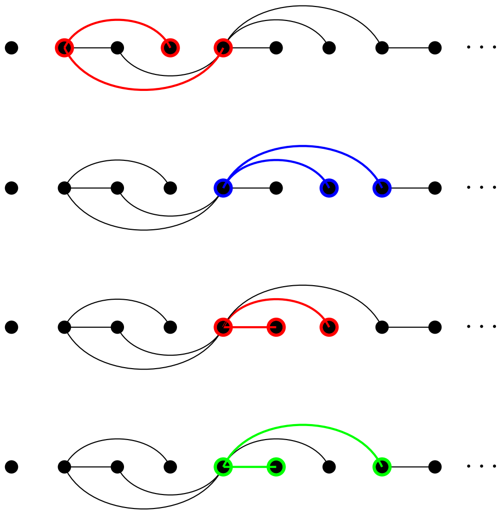


Figure: A copy of A in B

More copies of A in B



Different Types of Colorings on Graphs

Let G be a given graph.

Vertex Colorings: The vertices in G are colored.

Edge Colorings: The edges in G are colored.

Colorings of Triangles: All triangles in G are colored. (These may be thought of as hyperedges.)

Colorings of *n*-cycles: All *n*-cycles in G are colored.

Colorings of A: Given a finite graph A, all copies of A which occur in G are colored.

Ramsey Theorem for Finite Ordered Graphs

Thm. (Nešetřil/Rödl 1977/83) For any finite ordered graphs A and B such that $\mathrm{A} \leq \mathrm{B}$, there is a finite ordered graph C such that for each coloring of all the copies of A in C into red and blue, there is a $\mathrm{B}' \leq \mathrm{C}$ which is a copy of B such that all copies of A in B' have the same color.

In symbols, given any $f : \binom{\mathrm{C}}{\mathrm{A}} \rightarrow 2$, there is a $\mathrm{B}' \in \binom{\mathrm{C}}{\mathrm{B}}$ such that f takes only one color on all members of $\binom{\mathrm{B}'}{\mathrm{A}}$.

The Random Graph

The random graph is the graph on infinitely many nodes such that for each pair of nodes, there is a 50-50 chance that there is an edge between them.

This is often called the Rado graph since it was constructed by Rado, and is denoted by $\mathcal{R}$.

The random graph is

1. the Fraïssé limit of the Fraïssé class of all finite graphs.
2. universal for countable graphs: Every countable graph embeds into $\mathcal{R}$.
3. homogeneous: Every isomorphism between two finite subgraphs in $\mathcal{R}$ is extendible to an automorphism of $\mathcal{R}$.

Vertex Colorings in $\mathcal{R}$

Thm. (Folklore) Given any coloring of vertices in $\mathcal{R}$ into finitely many colors, there is a subgraph $\mathcal{R}' \leq \mathcal{R}$ which is also a random graph such that the vertices in $\mathcal{R}'$ all have the same color.

Edge Colorings in $\mathcal{R}$

Thm. (Pouzet/Sauer 1996) Given any coloring of the edges in $\mathcal{R}$ into finitely many colors, there is a subgraph $\mathcal{R}' \leq \mathcal{R}$ which is also a random graph such that the edges in $\mathcal{R}'$ take no more than two colors.

Can we get down to one color?

No!

Colorings of Copies of Any Finite Graph in $\mathcal{R}$

Thm. (Sauer 2006) Given any finite graph A, there is a finite number $n(\mathrm{A})$ such that the following holds:

For any $l \geq 1$ and any coloring of all the copies of A in $\mathcal{R}$ into l colors, there is a subgraph $\mathcal{R}' \leq \mathcal{R}$, also a random graph, such that the set of copies of A in $\mathcal{R}'$ take on no more than $n(\mathrm{A})$ colors.

We say that the big Ramsey degrees for $\mathcal{R}$ are finite, because we can find a copy of the whole infinite graph $\mathcal{R}$ in which all copies of A have at most some bounded number of colors.

The Main Steps in Sauer's Proof

Proof outline:

1. Graphs can be coded by trees.
2. Only diagonal trees need be considered.
3. Each diagonal tree can be enveloped in certain strong trees, called their envelopes.
4. Given a fixed diagonal tree A, if its envelopes are of form $2^{\leq k}$, then each strong subtree of $2^{<\omega}$ isomorphic to $2^{\leq k}$ contains a unique copy of A. Color the strong subtree by the color of its copy of A.
5. Apply Milliken's Theorem to the coloring on the strong subtrees of $2^{<\omega}$ of the form $2^{\leq k}$.
6. The number of isomorphism types of diagonal trees coding A gives the number $n(\mathrm{A})$.

Strong Trees

A tree $T \subseteq 2^{<\omega}$ is a strong tree if there is a set of levels $L \subseteq \mathbb{N}$ such that each node in T has length in L, and every non-terminal node in T branches.

Each strong tree is either isomorphic to $2^{<\omega}$ or to $2^{\leq k}$ for some finite k.

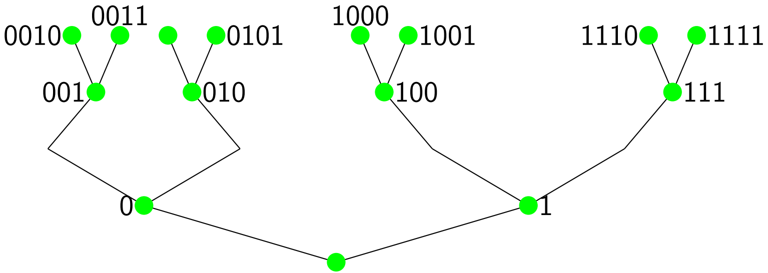
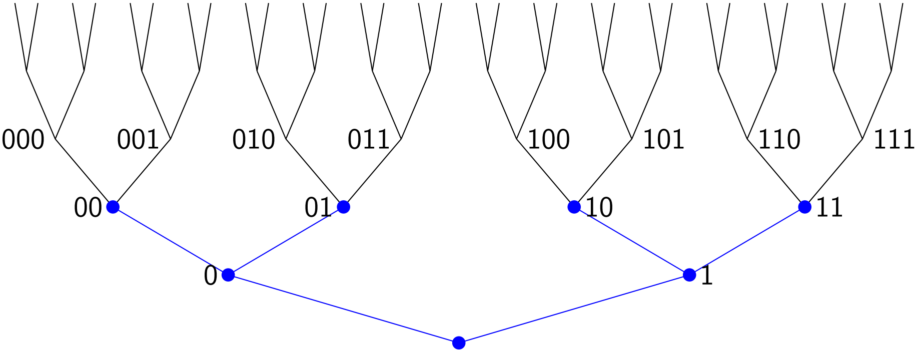
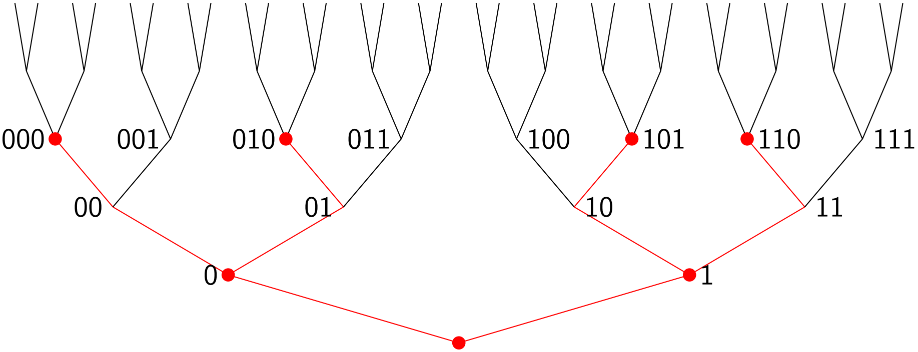


Figure: A strong subtree isomorphic to $2^{\leq 3}$

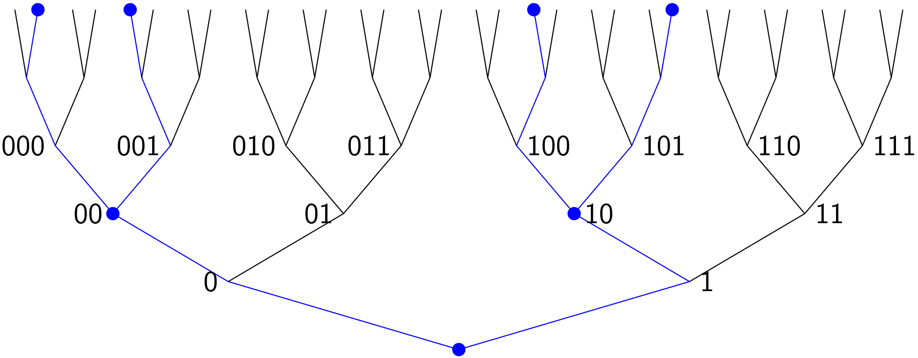
Strong Subtree $\cong 2^{\leq 2}$, Ex. 1



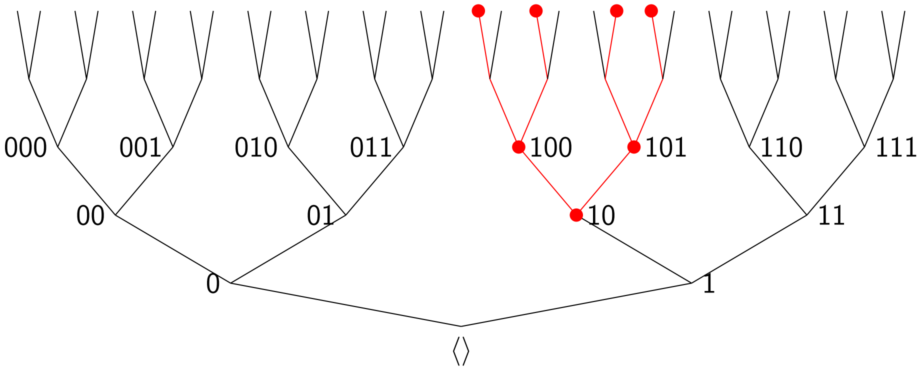
Strong Subtree $\cong 2^{\leq 2}$, Ex. 2



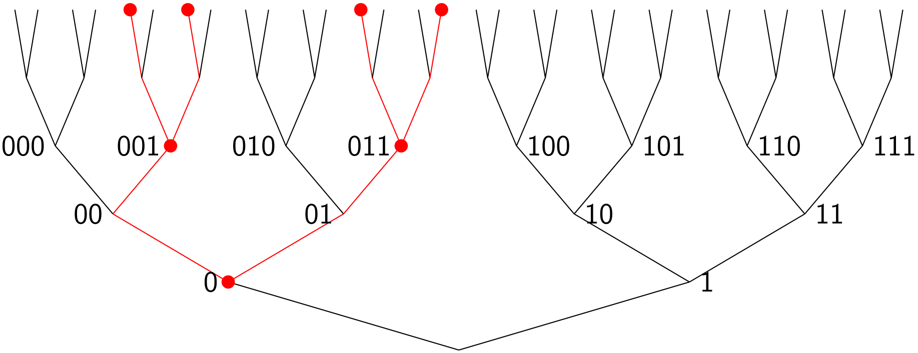
Strong Subtree $\cong 2^{\leq 2}$, Ex. 3



Strong Subtree $\cong 2^{\leq 2}$, Ex. 4



Strong Subtree $\cong 2^{\leq 2}$, Ex. 5



Milliken's Theorem (1981)

Let T be an infinite strong tree, $k \geq 0$, and let f be a coloring of all the finite strong subtrees of T which are isomorphic to $2^{\leq k}$.

Then there is an infinite strong subtree $S \subseteq T$ such that all copies of $2^{\leq k}$ in S have the same color.

Remark 1. For $k = 0$, the coloring is on the nodes of the tree T.

Remark 2. Milliken's Theorem builds on the Halpern-Läuchli Theorem for colorings of products of level sets of finitely many trees.

Ramsey theory for homogeneous structures has seen increased activity in recent years.

A homogeneous structure $\mathcal{S}$ which is a Fraïssé limit of some Fraïssé class $\mathcal{K}$ of finite structures is said to have finite big Ramsey degrees if for each $A \in \mathcal{K}$ there is a finite number $n(A)$ such that for any coloring of all copies of A in $\mathcal{S}$ into finitely many colors, there is a substructure $\mathcal{S}'$ which is isomorphic to $\mathcal{S}$ such that all copies of A in $\mathcal{S}'$ take on no more than $n(A)$ colors.

Question. Which homogeneous structures have finite big Ramsey degrees?

Question. What if some irreducible substructure is omitted?

Triangle-free graphs

A graph G is triangle-free if no copy of a triangle occurs in G.

In other words, given any three vertices in G, at least two of the vertices have no edge between them.

Finite Ordered Triangle-Free Graphs have Ramsey Property

Theorem. (Nešetřil-Rödl 1977/83) Given finite ordered triangle-free graphs $\mathrm{A} \leq \mathrm{B}$, there is a finite ordered triangle-free graph C such that for any coloring of the copies of A in C, there is a copy $\mathrm{B}' \in \binom{\mathrm{C}}{\mathrm{B}}$ such that all copies of A in B' have the same color.

The Universal Triangle-Free Graph

The universal triangle-free graph $\mathcal{H}_3$ is the triangle-free graph on infinitely many vertices into which every countable triangle-free graph embeds.

The universal triangle-free graph is also homogeneous: Any isomorphism between two finite subgraphs of $\mathcal{H}_3$ extends to an automorphism of $\mathcal{H}_3$.

$\mathcal{H}_3$ is the Fraïssé limit of the Fraïssé class $\mathcal{K}_3$ of finite ordered triangle-free graphs.

The universal triangle-free graph was constructed by Henson in 1971. Henson also constructed universal k-clique-free graphs for each $k \geq 3$.

Vertex Colorings

Theorem. (Henson 1971) Given any coloring of the vertices of $\mathcal{H}_3$ into red and blue, either there is a copy of $\mathcal{H}_3$ with only red vertices or else there is a subgraph with only blue vertices into which each finite triangle-free graph embeds.

Theorem. (Komjáth/Rödl 1986) For each coloring of the vertices of $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H}' \leq \mathcal{H}_3$ which is also universal triangle-free in which all vertices have the same color.

Edge Colorings

Theorem. (Sauer 1998) For each coloring of the edges of $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H}' \leq \mathcal{H}_3$ which is also universal triangle-free such that all edges in $\mathcal{H}$ have at most 2 colors.

This is best possible for edges.

Are the big Ramsey degrees for $\mathcal{H}_3$ finite?

What about colorings of finite triangle-free graphs in general?

Are the big Ramsey degrees for $\mathcal{H}_3$ finite?

That is, given any finite triangle-free graph A, is there a number $n(\mathrm{A})$ such that for any l and any coloring of the copies of A in $\mathcal{H}_3$ into l colors, there is a subgraph $\mathcal{H}$ of $\mathcal{H}_3$ which is also universal triangle-free, and in which all copies of A take on no more than $n(\mathrm{A})$ colors?

$\mathcal{H}_3$ has Finite Big Ramsey Degrees

Theorem. (D.) For each finite triangle-free graph A, there is a number $n(\mathrm{A})$ such that for any coloring of the copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H}' \leq \mathcal{H}_3$ which is also universal triangle-free such that all copies of A in $\mathcal{H}'$ take no more than $n(A)$ colors.

Structure of Proof that $\mathcal{H}_3$ has finite big Ramsey degrees

(1) Develop new notion of strong triangle-free tree coding $\mathcal{H}_3$.

These trees have distinguished coding nodes coding the vertices of the graph and branch as much as possible without any branch coding a triangle.

(2) Develop space of strong coding trees, analogous to the Milliken topological Ramsey space of strong trees.

This includes criteria on which subtrees are extendable to trees coding $\mathcal{H}_3$ (Parallel 1's Criterion).

(3) Prove a Ramsey theorem for finite subtrees of $\mathbb{T}$ satisfying the Parallel 1's Criterion.

The proof uses forcing but is in ZFC. First prove analogues of Halpern-Läuchli; then prove analogues of Milliken.

Structure of Proof that $\mathcal{H}_3$ has finite big Ramsey degrees

(4) Find the correct notion of envelope to extend a diagonal tree coding a finite triangle-free graph to a tree satisfying the Parallel 1's Criterion.

(5) Given a finite triangle-free graph G, transfer colorings from diagonal trees coding G to their envelopes. Apply the Ramsey theorem to obtain a strong coding tree $T \subseteq \mathbb{T}$ with one color for each of the finitely many possible similarity types of envelopes.

(6) Take a diagonal subtree of $D \subseteq T$ which codes $\mathcal{H}_3$ and is homogeneous for each type coding G along with a collection $W \subseteq T$ of witnessing nodes which are used to construct envelopes. Obtain the finite big Ramsey degree for G.

Rem. The space of strong coding trees satisfies all of Todorcevic's axioms for topological Ramsey spaces except **A.3 (2)**.

Trees can Code Graphs

Let A be a graph.

Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

The n-th **coding node** t_n in $2^{<\omega}$ codes v_n.

For each pair $i < n$,

$$v_n \; E \; v_i \Leftrightarrow t_n(|t_i|) = 1.$$

Finite strong triangle-free trees

Finite strong triangle-free trees are trees which code a triangle-free graph and which branch as much as possible, subject to the

Triangle-Free Extension Criterion: A node t at the level of the n-th coding node t_n extends right if and only if t and t_n have no parallel 1's.

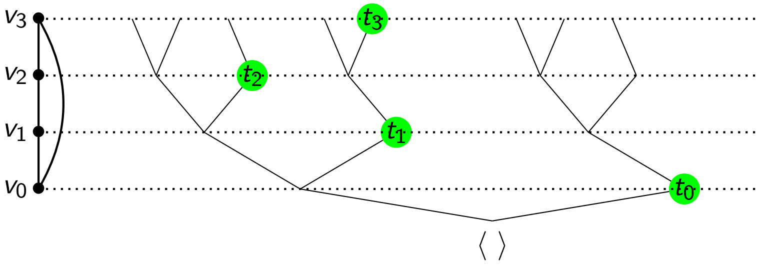
Every node always extends left.

Unlike Sauer's approach, we build the coding nodes into our trees; our language is the language of trees plus a unary predicate to indicate coding nodes.

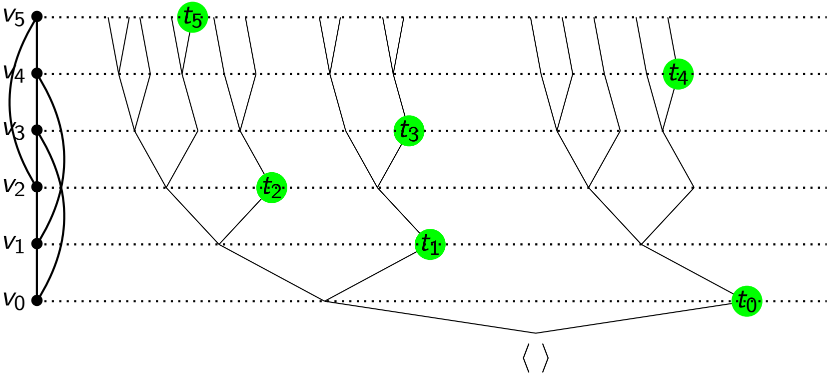
Building a strong triangle-free tree $\mathbb{T}^*$ coding $\mathcal{H}_3$

Let $\langle F_i : i < \omega \rangle$ be a listing of all finite subsets of $\mathbb{N}$ such that each set repeats infinitely many times.

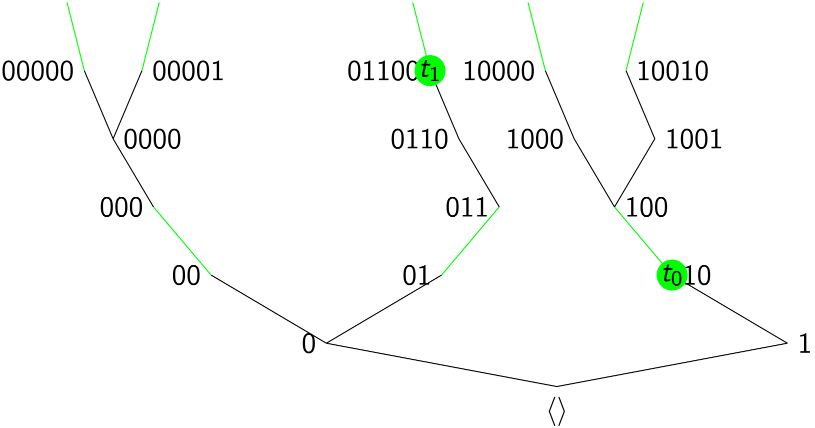
Alternate taking care of requirement F_i and taking care of density requirement for the coding nodes.



Building a strong triangle-free tree $\mathbb{T}^*$ to code $\mathcal{H}_3$



Skew $\mathbb{T}^*$ to construct the Strong Coding Tree $\mathbb{T}$



Extending finite subtrees to strong coding trees

A subtree $S \subseteq \mathbb{T}$ satisfies the Parallel 1's Criterion if whenever two nodes $s, t \in S$ have parallel 1's, there is a coding node in S witnessing this.

Every finite initial subtree of a strong coding tree satisfies the Parallel 1's Criterion.

Extension Lemma. Any maximally splitting finite subtree $U \subseteq \mathbb{T}$ satisfying the Parallel 1's Criterion can be extended inside $\mathbb{T}$ to a strong triangle-free subtree $T \subseteq \mathbb{T}$ which also codes $\mathcal{H}_3$.

Fact. If $T \subseteq \mathbb{T}$ is maximally splitting, satisfies the Parallel 1's Criterion, and the coding nodes in T are dense in T, then T codes $\mathcal{H}_3$.

Ramsey theory for strong coding trees

Theorem. (D.) Let $\mathbb{T}$ be a strong coding tree. Let A be a finite triangle-free tree satisfying the Parallel 1's Criterion, and let c color all the copies of A in $\mathbb{T}$ into finitely many colors.

Then there is a subtree T of $\mathbb{T}$ which is isomorphic to $\mathbb{T}$ (hence codes $\mathcal{H}_3$) such that all copies of A in T have the same color.

Remark. This is the analogue of Milliken's Theorem for the new setting of strong triangle-free trees with distinguished coding nodes.

copy of A has a precise definition.

Halpern-Läuchli analogues: level set extensions

There are two types of level sets to consider.

Case (a). End-extensions of a fixed finite tree to a new level with a splitting node.

Case (b) End-extensions of a fixed finite tree to a new level with a coding node.

We do different forcings for the two cases and obtain the following theorem.

Halpern-Läuchli analogues: level set extensions

Thm. (D.) Let A be a fixed finite subtree of $\mathbb{T}$ satisfying the Parallel 1's Criterion, and let X be a level set extension of A in $\mathbb{T}$ so that $A \cup X$ satisfies the Parallel 1's Criterion. Let B be the minimal initial subtree of $\mathbb{T}$ containing A

Case (a). (X contains a splitting node) Given a coloring c of all extensions Y of A in $\mathbb{T}$ such that $A \cup Y \cong A \cup X$ into two colors, there is a strong coding tree $T \subseteq \mathbb{T}$ extending B so that each extension of A to a copy of X in T has the same color.

Case (b). (X contains a coding node) Given a coloring c of all extensions Y of A in $\mathbb{T}$ such that $A \cup Y \cong A \cup X$ into two colors, there is a strong coding tree $T \subseteq \mathbb{T}$ extending B so that T is end-homogeneous above each minimal copy of X extending A in T.

Homogenizing Case (b)

If X contains a coding node, to homogenize over the end-homogeneity, we do a third type of forcing (Case (c)) plus much fusion to obtain

Thm. (D.) Let A be a fixed finite subtree of $\mathbb{T}$ satisfying the Parallel 1's Criterion, and let X be a level set extension of A in $\mathbb{T}$ containing a coding node so that $A \cup X$ satisfies the Parallel 1's Criterion. Let B be the minimal initial subtree of $\mathbb{T}$ containing A.

Given a coloring c of all extensions Y of A in $\mathbb{T}$ such that $A \cup Y \cong A \cup X$ into two colors, there is a strong coding tree $T \subseteq \mathbb{T}$ extending B such that each such Y extending A in T has the same color.

The forcing ideas

The simplest of the three cases is Case (a) when the level set X extending A has a splitting node.

Let T be a strong coding tree.

List the maximal nodes of A^+ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in X extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \le d$.

Fix κ large enough so that $\kappa \to (\aleph_1)^{2d+2}_{\aleph_0}$ holds.

The forcing for Case (a)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

$q \leq p$ if and only if either

1. $l_q = l_p$ and $q \supseteq p$ (so also $\vec{\delta}_q \supseteq \vec{\delta}_p$); or else
2. $l_q > l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and
   (i) $q(d) \supset p(d)$, and for each $\delta \in \vec{\delta}_p$ and $i < d$, $q(i,\delta) \supset p(i,\delta)$;
   (ii) Whenever $(\alpha_0, \dots, \alpha_{d-1})$ is a strictly increasing sequence in $(\vec{\delta}_p)^d$ and $\{p(i,\alpha_i) : i < d\} \cup \{p(d)\} \in \mathrm{Ext}_T(A,X)$, then also $\{q(i,\alpha_i) : i < d\} \cup \{q(d)\} \in \mathrm{Ext}_T(A,X)$.

Remarks. The forcing is used to do an unbounded search for the next level and sets of nodes on that level isomorphic to X which have the same color, but no generic extension is actually used.

These forcings are not simply Cohen forcings; the partial ordering is stronger in order to guarantee that the new levels we obtain by forcing are extendible inside $\mathbb{T}$ to another strong coding tree. The Parallel $\mathbf{1}$'s Criterion is necessary.

Ramsey theorem for finite trees

Thm. (D.) Let A be a finite triangle-free tree satisfying the Parallel 1's Criterion, and let c be a coloring of all copies of A in a strong coding tree $\mathbb{T}$.

Then there is a strong coding tree $T \subseteq \mathbb{T}$ in which all strict copies of A in T have the same color.

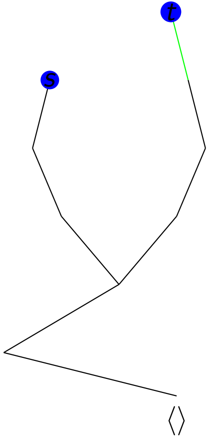
Rem. In particular, given a fixed initial segment A of $\mathbb{T}$, the collection of all strict copies of A in $\mathbb{T}$ has the Ramsey property.
(There is of course a precise definition of strict copy.)

Envelopes with the Parallel 1's Criterion

Roughly, an envelope of a finite triangle-free tree A is a minimal extension to a tree $E(A)$ satisfying the Parallel 1's Criterion.

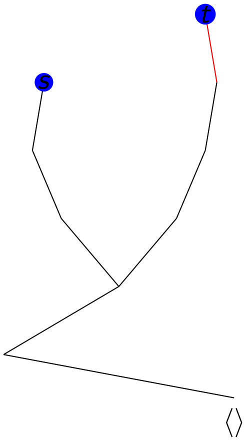
(There is of course a precise definition of envelope.)

A tree A coding a non-edge

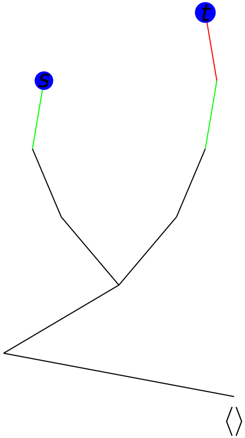


This satisfies the Parallel 1's Criterion, so $E(A) = A$.

Another tree B coding a non-edge

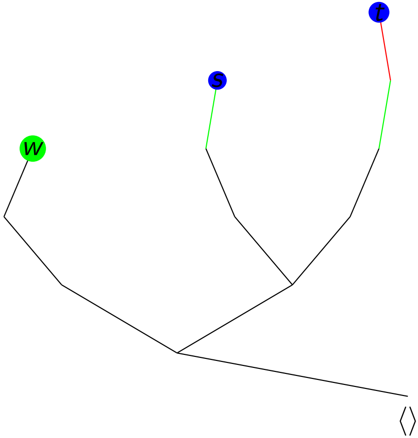


Another tree B coding a non-edge



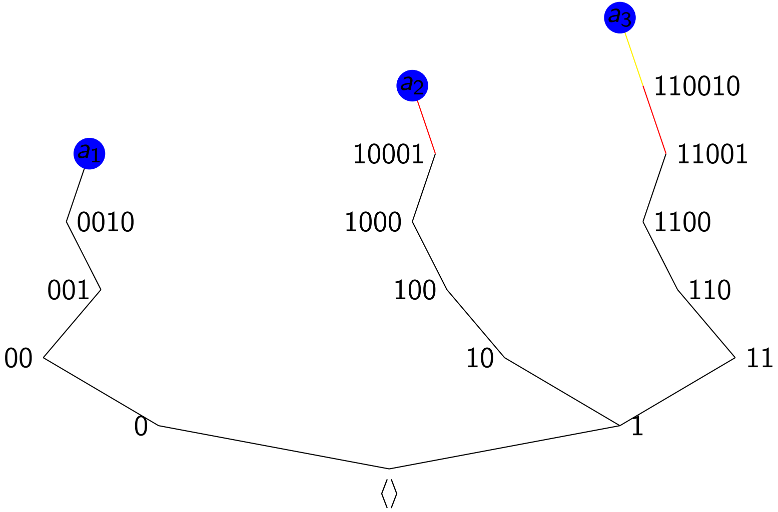
This tree has parallel 1's which are not witnessed by a coding node.

An Envelope $E(B)$

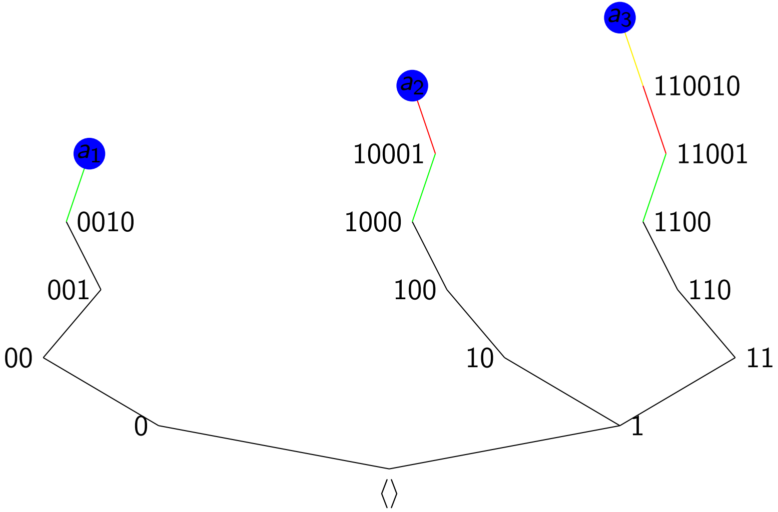


This satisfies the Parallel 1's Criterion.

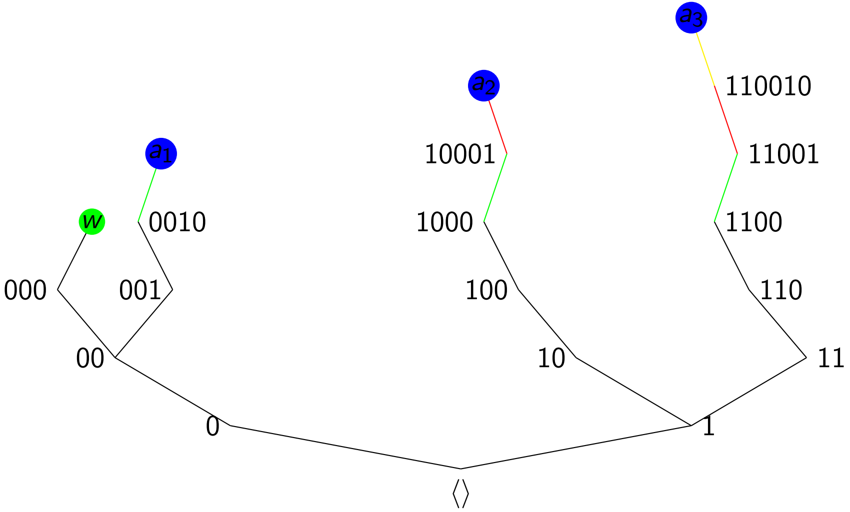
Example: X coding three vertices with no edges



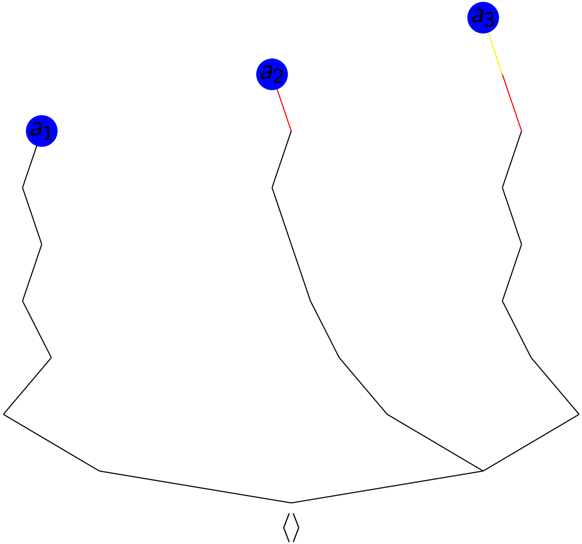
X has some parallel 1's not witnessed by a coding node



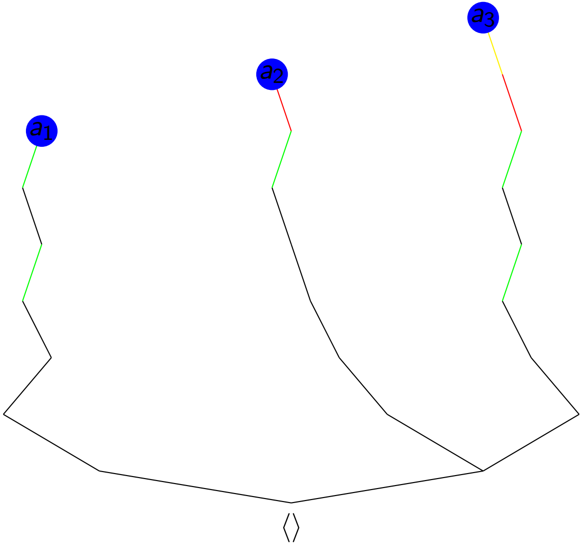
An Envelope $E(X)$



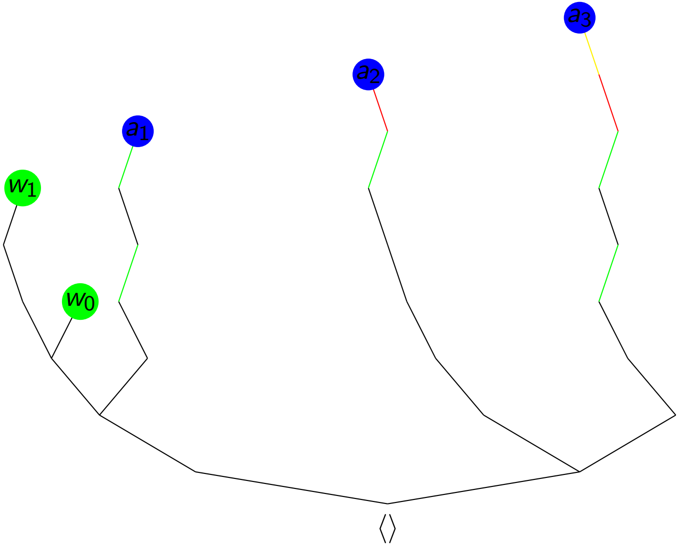
Example: Y coding three vertices with no edges



Y has some parallel 1's not witnessed by a coding node



A Strict Envelope $E(Y)$



Finite Big Ramsey Degrees for $\mathcal{H}_3$

The envelopes are actually a means of obtaining a cleaner Ramsey theorem, where the coloring simply depends on the strict similarity type of a finite triangle-free tree.

Thm. (D.) Given a finite subtree A of a strong coding tree $\mathbb{T}$, and given a finite coloring of all strictly similar copies of A in $\mathbb{T}$, there is a strong coding tree $T \subseteq \mathbb{T}$ in which each strictly similar copy of A in T has the same color.

Rem. These are providing the finite bound for the big Ramsey degree of a fixed finite triangle-free graph. Two trees are strictly similar if their strict envelopes are isomorphic as trees with coding nodes.

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Thank you!