On the finite big Ramsey degrees for the universal triangle-free graph: A progress report

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Arctic Set Theory III, January 2017

Graphs and Ordered Graphs

Graphs are sets of vertices with edges between some of the pairs of vertices.

An ordered graph is a graph whose vertices are linearly ordered.

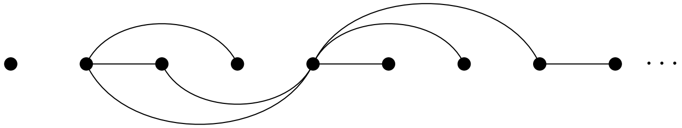


Figure: An ordered graph B

Embeddings of Graphs

An ordered graph A embeds into an ordered graph B if there is a one-to-one mapping of the vertices of A into some of the vertices of B such that each edge in A gets mapped to an edge in B, and each non-edge in A gets mapped to a non-edge in B.

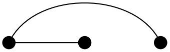


Figure: A

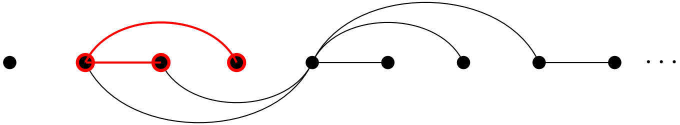
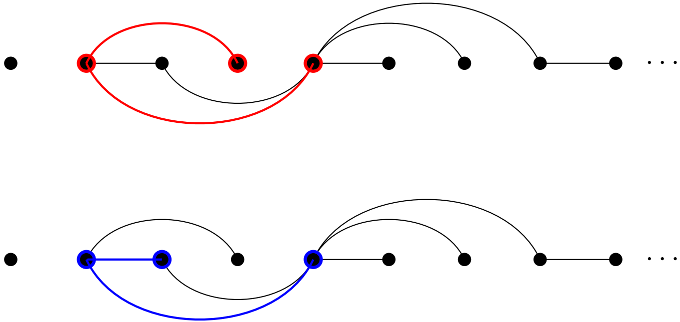
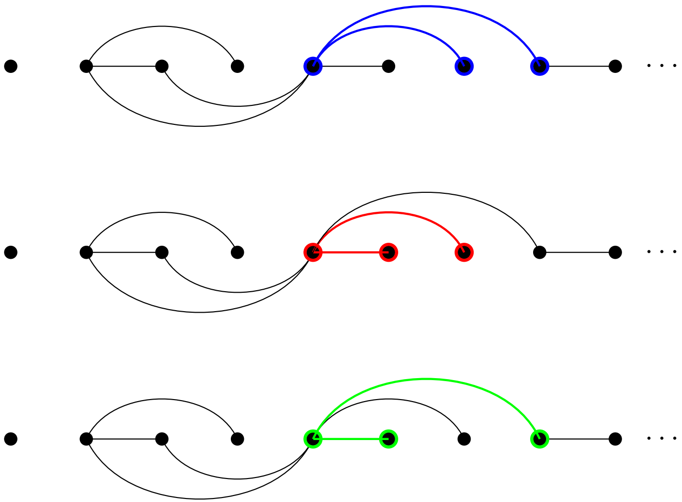


Figure: A copy of A in B

More copies of A into B



Still more copies of A into B



Different Types of Colorings on Graphs

Let G be a given graph.

Vertex Colorings: The vertices in G are colored.

Edge Colorings: The edges in G are colored.

Colorings of Triangles: All triangles in G are colored. (These may be thought of as hyperedges.)

Colorings of *n*-cycles: All *n*-cycles in G are colored.

Colorings of A: Given a finite graph A, all copies of A which occur in G are colored.

Ramsey Theorem for Finite Ordered Graphs

Thm. (Nešetřil/Rödl) For any finite ordered graphs A and B such that $\mathrm{A} \le \mathrm{B}$, there is a finite ordered graph C such that for each coloring of all the copies of A in C into red and blue, there is a $\mathrm{B}' \le \mathrm{C}$ which is a copy of B such that all copies of A in B' have the same color.

In symbols, given any $f : \binom{\mathrm{C}}{\mathrm{A}} \to 2$, there is a $\mathrm{B}' \in \binom{\mathrm{C}}{\mathrm{B}}$ such that f takes only one color on all members of $\binom{\mathrm{B}'}{\mathrm{A}}$.

The Random Graph

The random graph is the graph on infinitely many nodes such that for each pair of nodes, there is a 50-50 chance that there is an edge between them.

This is often called the Rado graph since it was constructed by Rado, and is denoted by $\mathcal{R}$.

The random graph is

1. the Fraïssé limit of the Fraïssé class of all countable graphs.
2. universal for countable graphs: Every countable graph embeds into $\mathcal{R}$.
3. homogeneous: Every isomorphism between two finite subgraphs in $\mathcal{R}$ is extendible to an automorphism of $\mathcal{R}$.

Vertex Colorings in $\mathcal{R}$

Thm. (Folklore) Given any coloring of vertices in $\mathcal{R}$ into finitely many colors, there is a subgraph $\mathcal{R}' \leq \mathcal{R}$ which is also a random graph such that the vertices in $\mathcal{R}'$ all have the same color.

Edge Colorings in $\mathcal{R}$

Thm. (Pouzet/Sauer) Given any coloring of the edges in $\mathcal{R}$ into finitely many colors, there is a subgraph $\mathcal{R}' \leq \mathcal{R}$ which is also a random graph such that the edges in $\mathcal{R}'$ take no more than two colors.

Can we get down to one color?

No!

Colorings of Copies of Any Finite Graph in $\mathcal{R}$

Thm. (Sauer) Given any finite graph A, there is a finite number $n(\mathrm{A})$ such that the following holds:

For any $l \geq 1$ and any coloring of all the copies of A in $\mathcal{R}$ into l colors, there is a subgraph $\mathcal{R}' \leq \mathcal{R}$, also a random graph, such that the set of copies of A in $\mathcal{R}'$ take on no more than $n(\mathrm{A})$ colors.

In the jargon, we say that the big Ramsey degrees for $\mathcal{R}$ are finite, because we can find a copy of the whole infinite graph $\mathcal{R}$ in which all copies of A have at most some bounded number of colors.

The proof that this is best possible uses Ramsey theory on trees.

Strong Trees

A tree $T \subseteq 2^{<\omega}$ is a strong tree if there is a set of levels $L \subseteq \mathbb{N}$ such that each node in T has length in L, and every non-maximal node in T branches.

Each strong tree is either isomorphic to $2^{<\omega}$ or to $2^{\leq k}$ for some finite k.

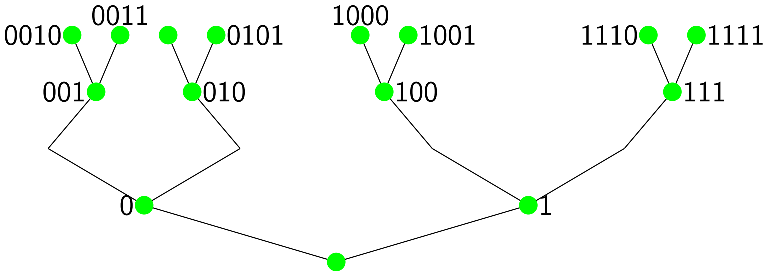
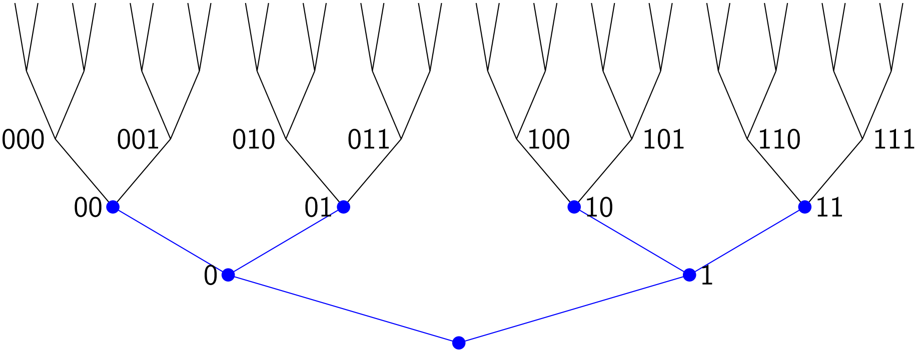
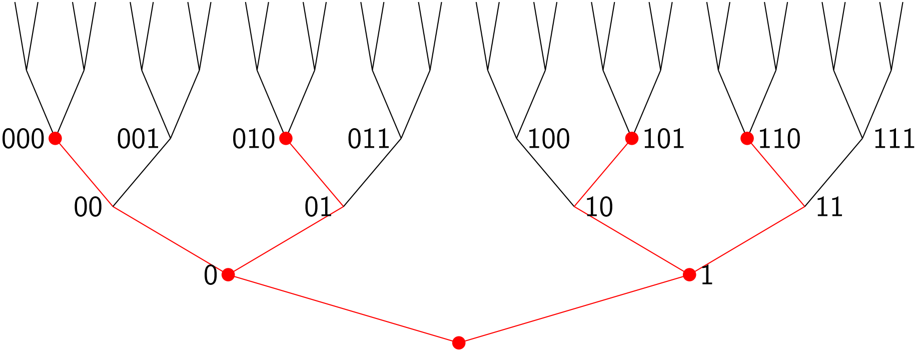


Figure: A strong subtree isomorphic to $2^{\leq 3}$

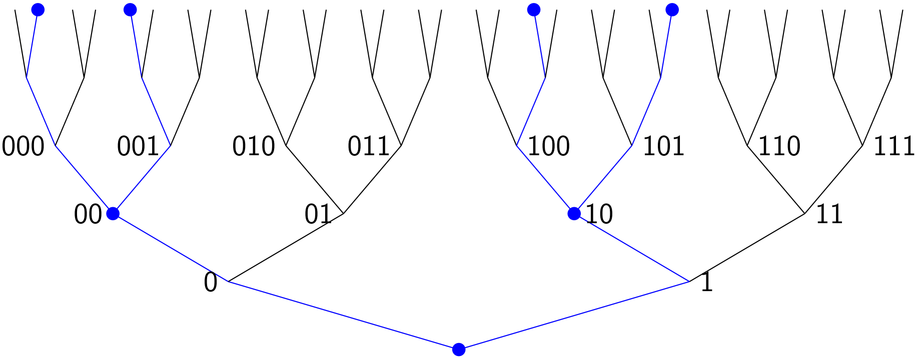
Strong Subtree $\cong 2^{\leq 2}$, Ex. 1



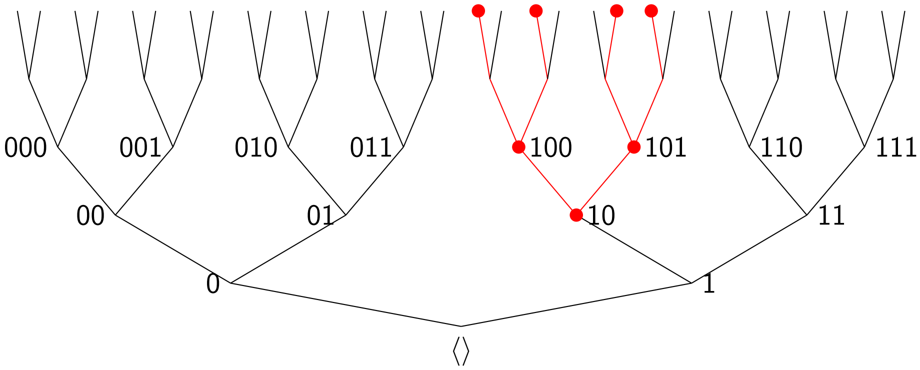
Strong Subtree $\cong 2^{\leq 2}$, Ex. 2



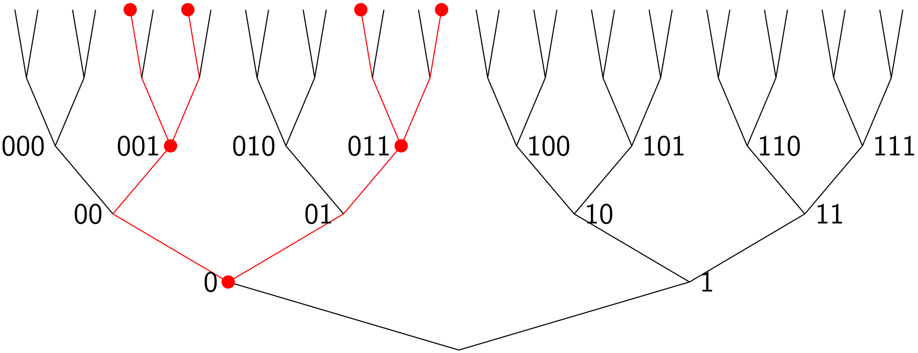
Strong Subtree $\cong 2^{\leq 2}$, Ex. 3



Strong Subtree $\cong 2^{\leq 2}$, Ex. 4



Strong Subtree $\cong 2^{\leq 2}$, Ex. 5



Milliken's Theorem

Let T be an infinite strong tree, $k \geq 0$, and let f be a coloring of all the finite strong subtrees of T which are isomorphic to $2^{\leq k}$.

Then there is an infinite strong subtree $S \subseteq T$ such that all copies of $2^{\leq k}$ in S have the same color.

Remark. For $k = 0$, the coloring is on the nodes of the tree T.

The Main Steps in Sauer's Proof

Proof outline:

1. Graphs can be coded by trees.
2. Only diagonal trees need be considered.
3. Each diagonal tree can be enveloped in certain strong trees, called their *envelopes*.
4. Given a fixed diagonal tree A, if its envelope is of form $2^{\leq k}$, then each strong subtree of $2^{<\omega}$ isomorphic to $2^{\leq k}$ contains a unique copy of A. Color the strong subtree by the color of its copy of A.
5. Apply Milliken's Theorem to the coloring on the strong subtrees of $2^{<\omega}$ of the form $2^{\leq k}$.
6. The number of isomorphism types of diagonal trees coding A gives the number $n(\mathrm{A})$.

Using Trees to Code Graphs

Let A be a graph.

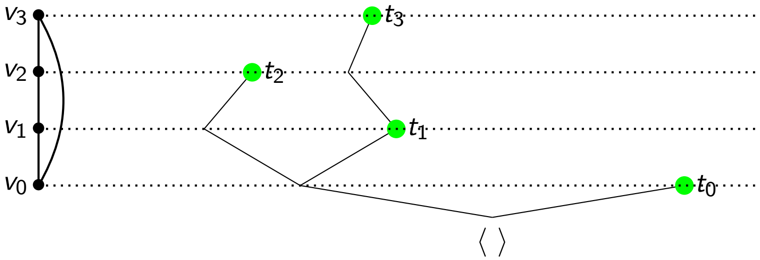
Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

The n-th **coding node** t_n in $2^{<\omega}$ codes v_n.

For each pair $i < n$,

$$v_n \; E \; v_i \Leftrightarrow t_n(|t_i|) = 1$$

A Tree Coding a 4-Cycle



Diagonal Trees Code Graphs

A tree T is diagonal if there is at most one meet or terminal node per level.

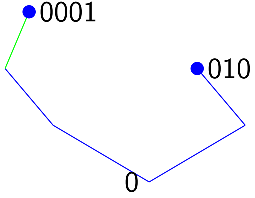


Figure: A diagonal tree D coding an edge between two vertices

Every graph can be coded by the terminal nodes of a diagonal tree.
Moreover, there is a diagonal tree which codes $\mathcal{R}$.

Strong Tree Envelopes of Diagonal Trees

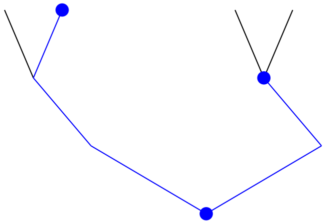


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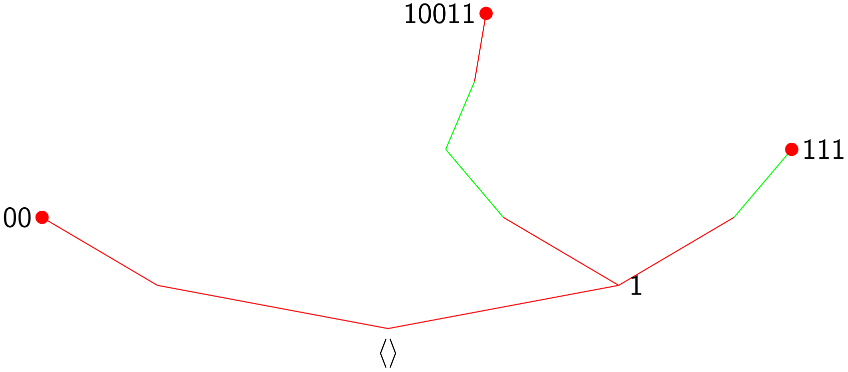
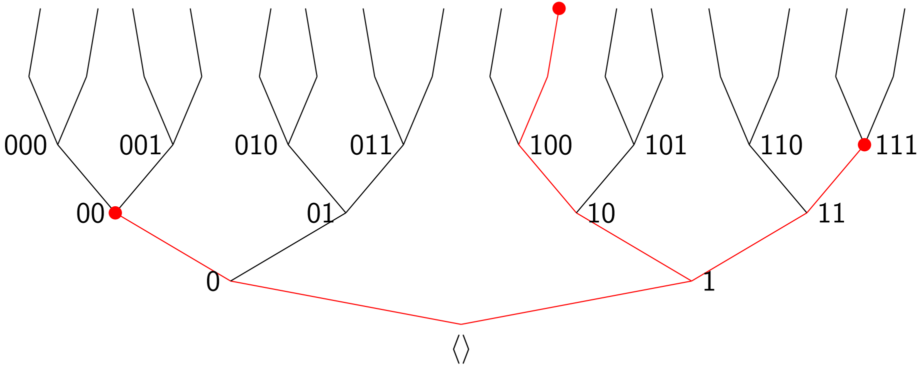
Figure: The strong tree enveloping D

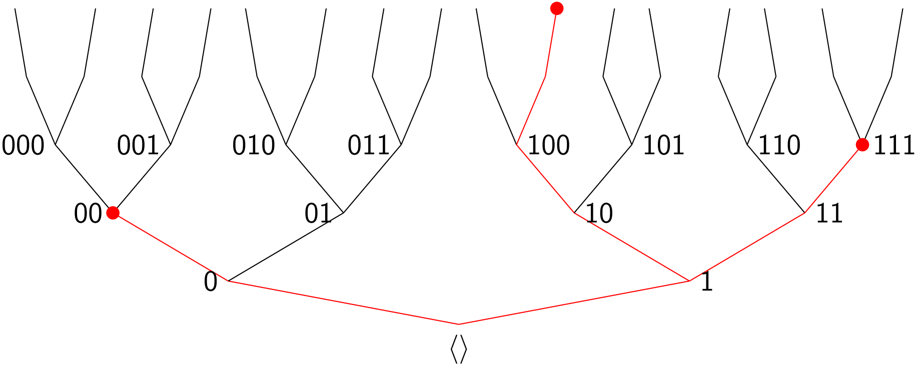
Strongly Diagonal Tree



Strongly Diagonal Tree and Subtree Envelope 1



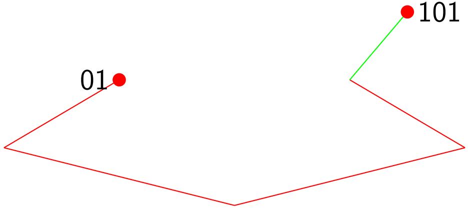
Strongly Diagonal Tree and Subtree Envelope 2



The Big Ramsey Degrees for the Random Graph

Theorem. (Sauer) The Ramsey degree for a given finite graph A in the Rado graph is the number of different isomorphism types of diagonal trees coding A.

There are exactly two types of diagonal trees coding an edge. The tree D a few slides ago, and the following type:



Ramsey theory for homogeneous structures has seen increased activity in recent years.

A homogeneous structure $\mathcal{S}$ which is a Fraïssé limit of some Fraïssé class $\mathcal{K}$ of finite structures is said to have finite big Ramsey degrees if for each $A \in \mathcal{K}$ there is a finite number $n(A)$ such that for any coloring of all copies of A in $\mathcal{S}$ into finitely many colors, there is a substructure $\mathcal{S}'$ which is isomorphic to $\mathcal{S}$ such that all copies of A in $\mathcal{S}'$ take on no more than $n(A)$ colors.

Question. Which homogeneous structures have finite big Ramsey degrees?

Question. What if some irreducible substructure is omitted?

Triangle-free graphs

A graph G is triangle-free if no copy of a triangle occurs in G.

In other words, given any three vertices in G, at least two of the vertices have no edge between them.

Finite Ordered Triangle-Free Graphs have Ramsey Property

Theorem. (Nešetřil-Rödl) Given finite ordered triangle-free graphs $\mathrm{A} \leq \mathrm{B}$, there is a finite ordered triangle-free graph C such that for any coloring of the copies of A in C, there is a copy $\mathrm{B}' \in \binom{\mathrm{C}}{\mathrm{B}}$ such that all copies of A in B' have the same color.

The Universal Triangle-Free Graph

The universal triangle-free graph $\mathcal{H}_3$ is the triangle-free graph on infinitely many vertices into which every countable triangle-free graph embeds.

The universal triangle-free graph is also homogeneous: Any isomorphism between two finite subgraphs of $\mathcal{H}_3$ extends to an automorphism of $\mathcal{H}_3$.

$\mathcal{H}_3$ is the Fraïssé limit of the Fraïssé class $\mathcal{K}_3$ of finite ordered triangle-free graphs.

The universal triangle-free graph was constructed by Henson in 1971. Henson also constructed universal k-clique-free graphs for each $k \geq 3$.

Vertex and Edge Colorings

Theorem. (Komjáth/Rödl) For each coloring of the vertices of $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H}' \leq \mathcal{H}_3$ which is also universal triangle-free in which all vertices have the same color.

Theorem. (Sauer) For each coloring of the edges of $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H}' \leq \mathcal{H}_3$ which is also universal triangle-free such that all edges in $\mathcal{H}$ have at most 2 colors.

This is best possible for edges.

Are the big Ramsey degrees for $\mathcal{H}_3$ finite?

That is, given any finite triangle-free graph A, is there a number $n(\mathrm{A})$ such that for any l and any coloring of the copies of A in $\mathcal{H}_3$ into l colors, there is a subgraph $\mathcal{H}$ of $\mathcal{H}_3$ which is also universal triangle-free, and in which all copies of A take on no more than $n(\mathrm{A})$ colors?

Three main obstacles:

1. There is no natural *sibling* of $\mathcal{H}_3$. ($\mathcal{R}$ and the graph coded by $2^{<\omega}$ are bi-embeddable and Sauer's proof relied strongly on this.)
2. There was no known useful way of coding $\mathcal{H}_3$ into a tree.
3. There was no analogue of Milliken's Theorem for $\mathcal{H}_3$.

Even if one had all that, one would still need a new notion of envelope.

So, this is what we did.

$\mathcal{H}_3$ has Finite Big Ramsey Degrees

Theorem*. (D.) For each finite triangle-free graph A, there is a number $n(\mathrm{A})$ such that for any coloring of the copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H}' \leq \mathcal{H}_3$ which is also universal triangle-free such that all copies of A in $\mathcal{H}'$ take no more than $n(A)$ colors.

$*$ 4/5ths finished typing.

Structure of Proof

(1) Develop a notion of strong triangle-free trees coding triangle-free graphs.

These trees have special coding nodes coding the vertices of the graph and branch as much as possible without any branch coding a triangle (Triangle-Free and Maximal Extension Criteria).

(2) Construct a strong triangle-free tree $\mathbb{T}^*$ coding $\mathcal{H}_3$ with the coding nodes dense in $\mathbb{T}^*$.

(3) Stretch $\mathbb{T}^*$ to a diagonal strong triangle-free tree $\mathbb{T}$ densely coding $\mathcal{H}_3$.

(4) Many subtrees of $\mathbb{T}$ can be extended within the given tree to form another coding of $\mathcal{H}_3$. (Parallel 1's Criterion, Extension Lemma).

(5) Prove a Ramsey theorem for finite subtrees of $\mathbb{T}$ satisfying the Parallel 1's Criterion.

(The proof uses forcing but is in ZFC, extending the proof method of Harrington's forcing proof of the Halpern-Läuchli Theorem.)

(6) For each finite triangle-free graph G there are finitely many isomorphism types of subtrees A of $\mathbb{T}$ coding G.

(7) Find the correct notion of a triangle-free envelope $E(\mathrm{A})$.

(8) Transfer colorings from diagonal trees to their envelopes. Apply the Ramsey theorem.

(9) Take a diagonal subtree of $\mathbb{T}$ which codes $\mathcal{H}_3$ and is homogeneous for each *type* coding G along with a collection W of 'witnessing nodes' which are used to construct envelopes.

Building a strong triangle-free tree $\mathbb{T}^*$ to code $\mathcal{H}_3$

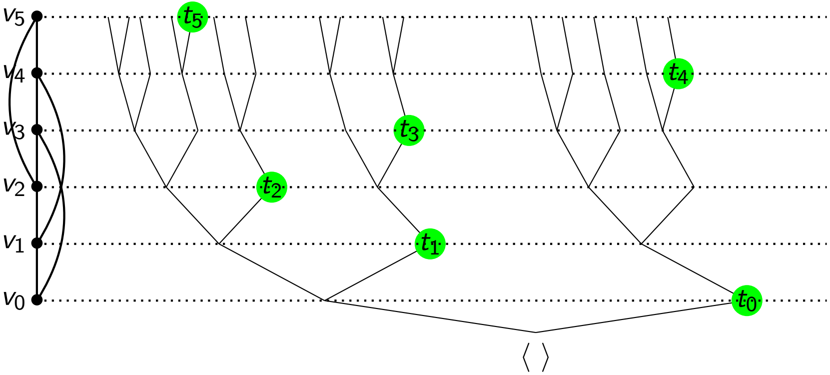
Let $\langle F_i : i < \omega \rangle$ be a listing of all finite subsets of $\mathbb{N}$ such that each set repeats infinitely many times.

Alternate taking care of requirement F_i and taking care of density requirement for the coding nodes.

Satisfy the Triangle Free Criterion: If s has the same length as a coding node t_n, and s and t_n have parallel 1's, then s can only extend left past t_n.

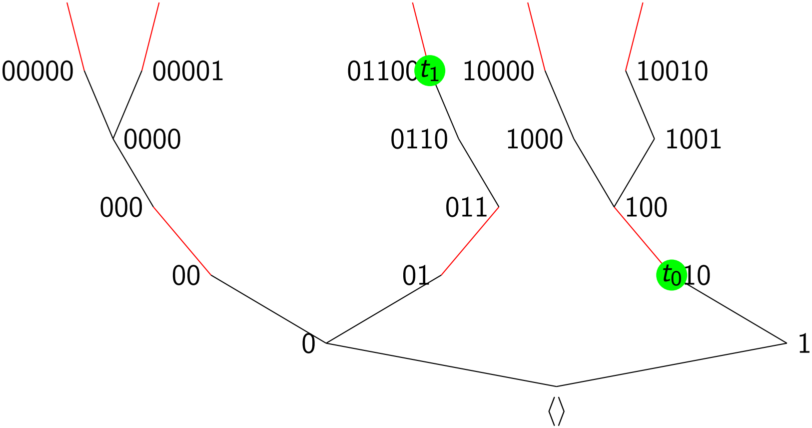
The TFC ensures that in each finite initial segment of $\mathbb{T}$, each node in $\mathbb{T}$ can be extended to a coding node without coding a triangle with any of the coding nodes already established.

Building a strong triangle-free $\mathbb{T}^*$ to code $\mathcal{H}_3$



$\mathbb{T}^*$ is a perfect tree.

Skew tree coding $\mathcal{H}_3$



A subtree $S \subseteq \mathbb{T}$ satisfies the Parallel 1's Criterion if whenever two nodes $s, t \in S$ have parallel 1's, there is a coding node in S *witnessing* this.

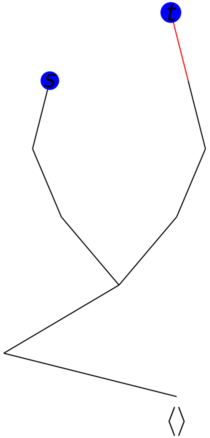
That is, if $s, t \in S$ and $s(l) = t(l) = 1$ for some l, then there is a coding node $c \in S$ such that $s(|c|) = t(|c|) = 1$ and the minimal l such that $s(l) = t(l) = 1$ has length between the longest splitting node in S below c and $|c|$.

This guarantees that a subtree of S of $\mathbb{T}$ can be extended in $\mathbb{T}$ to another strong tree coding $\mathcal{H}_3$. It is also necessary.

Strong Similarity Types of Trees Coding Graphs

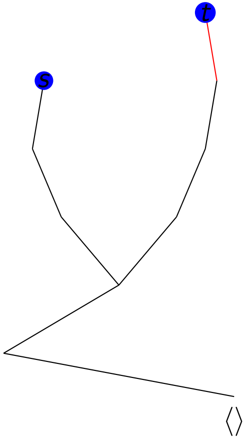
The similarity type is a strong notion of isomorphism, taking into account passing numbers at coding nodes, and when first parallel 1's occur.
This builds on Sauer's notion but adds a few more ingredients.

A tree coding a non-edge

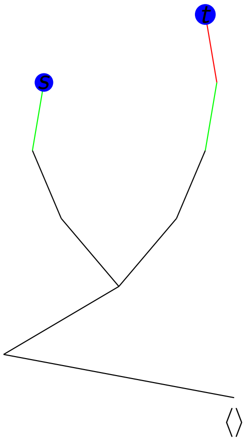


This is a strong similarity type satisfying the Parallel 1's Criterion.

Another tree coding a non-edge

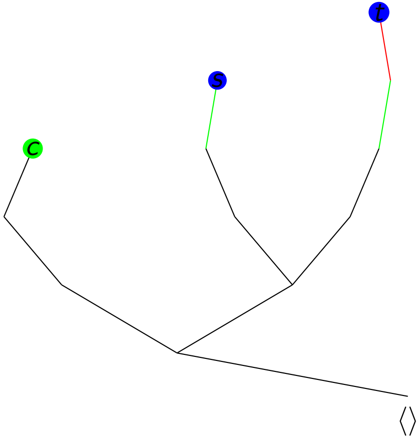


This is a strong similarity type not satisfying the Parallel 1's Criterion.



This tree has parallel 1's which are not witnessed by a coding node.

Its Envelope



This satisfies the Parallel 1's Criterion.

Ramsey theorem for strong triangle-free trees

Theorem. (D.) For each finite subtree A of $\mathbb{T}$ satisfying the Parallel 1's Criterion, for any coloring of all copies of A in $\mathbb{T}$ into finitely many colors, there is a subtree T of $\mathbb{T}$ which is isomorphic to $\mathbb{T}$ (hence codes $\mathcal{H}_3$) such that the copies of A in T have the same color.

Parallel 1's Criterion: A tree $A \subseteq \mathbb{T}$ satisfies the Parallel 1's Criterion if any two nodes with parallel 1's has a coding node witnessing that.

The proof uses three different forcings and much fusion

The simplest of the three cases is where we have a fixed tree A satisfying the Parallel 1's Criterion and a 1-level extension of A to some C which has one splitting node.

Fix T a strong triangle-free tree densely coding $\mathcal{G}_3$ and fix a copy of A in T. We are coloring all extensions of A in T which make a copy of C.

Let $d+1$ be the number of maximal nodes in C.

Fix κ large enough so that $\kappa \rightarrow (\aleph_1)^{2d+2}_{\aleph_0}$ holds.

The forcing for Case 1

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

$q \le p$ if and only if either

1. $l_q = l_p$ and $q \supseteq p$ (so also $\vec{\delta}_q \supseteq \vec{\delta}_p$); or else
2. $l_q > l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and
   - (i) $q(d) \supset p(d)$, and for each $\delta \in \vec{\delta}_p$ and $i < d$, $q(i,\delta) \supset p(i,\delta)$;
   - (ii) Whenever $(\alpha_0, \ldots, \alpha_{d-1})$ is a strictly increasing sequence in $(\vec{\delta}_p)^d$ and $\{p(i,\alpha_i) : i < d\} \cup \{p(d)\} \in \mathrm{Ext}_T(A, C)$, then also $\{q(i,\alpha_i) : i < d\} \cup \{q(d)\} \in \mathrm{Ext}_T(A, C)$.

The three types of forcings take care of

Case 1. End-extension of level sets to a new level with a splitting node. This gives homogeneity for end-extensions of A to next level.

Case 2. End-extension of level sets to a new level with a coding node. This gives end-homogeneity above a minimal extension of A with the correct passing numbers.

Case 3. Splitting predecessors and left branches if no splits of a level with a coding node. This allows to homogenize over the end-homogeneity in Case 2.

Eventually we obtain a strong triangle-free tree Scoding $\mathcal{H}_3$ such that every copy of C in S has the same color.

To finish, given a finite triangle-free graph G, there are only finitely many strong similarity types of trees coding G (with the coding nodes in the tree).

Each of these has a unique type of minimal extension to an envelope satisfying the Parallel 1's Criterion.

Apply the Ramsey theorem to these.

Obtain a finite bound for the big Ramsey degree of G inside $\mathcal{H}_3$.

Thanks!

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Most graphics in this talk were either made by or modified from codes made by Timothy Trujillo.