CREATURE FORCING AND TOPOLOGICAL RAMSEY SPACES

NATASHA DOBRINEN

Celebrating Alan Dow and his tours de force in set theory and topology

Abstract. This article introduces a line of investigation into connections between creature forcings and topological Ramsey spaces. Three examples of sets of pure candidates for creature forcings are shown to contain dense subsets which are actually topological Ramsey spaces. A new variant of the product tree Ramsey theorem is proved in order to obtain the pigeonhole principles for two of these examples.

1. Introduction

Connections between partition theorems and creature forcings have been known for some time. Partition theorems are used to establish various norm functions and to deduce forcing properties, for instance, properness. Conversely, creature forcings can give rise to new partition theorems, as seen, for instance, in [15]. Todorcevic pointed out to the author in 2008 that there are strong connections between creature forcings and topological Ramsey spaces deserving of a systematic investigation. The purpose of this note is to open up this line of research and provide some tools for future investigations.

In [15], Roslanowski and Shelah proved partition theorems for several broad classes of creature forcings. Their partition theorems have the following form: Given a creature forcing and letting $F_H$ denote the related countable set of finitary functions, for any partition of $F_H$ into finitely many pieces there is a pure candidate for which all finitary functions obtainable from it (the possibilities on the all creatures obtained from the pure candidate) reside in one piece of the partition. Their proofs proceed in a similar vein to Glazer’s proof of Hindman’s Theorem: Using the subcomposition function on pure candidates, they define an associative binary operation which gives rise to a semi-group on the set of creatures. Then they prove the existence of idempotent ultrafilters for this semi-group. As a consequence, they obtain the partition theorems mentioned above. In particular, assuming the Continuum Hypothesis, there is an ultrafilter on $F_H$ which is generated by pure candidates, analogously to ultrafilters on base set $[\omega]<\omega$ generated by infinite block sequences using Hindman’s Theorem.

In this article, we look at three specific examples of creature forcings from [15] and construct dense subsets of the collections of pure candidates which we prove form topological Ramsey spaces; that is, these dense subsets satisfy the Abstract Ellentuck Theorem: In the related exponential topology, every subset which has the

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property of Baire is Ramsey. As a corollary, we recover Roslanowski and Shelah’s partition theorems for these particular examples.

Showing that the Axiom A.4 (pigeonhole) holds for these forcings is quite related to, but in general not the same as, the partition theorems in [15]. However, for two of these examples, showing that there are dense subsets forming a topological Ramsey space is actually stronger, and the related partition theorems in [15] are recovered. For these two examples, the pigeonhole principle relies on a Ramsey theorem for unbounded finite products of finite sets, where exactly one of the sets in the product can be replaced with the collection of its \( k \)-sized subsets. This is proved in Theorem 3 in Section 3 building on work of Di Prisco, Llopis and Todorcevic in [3]. The method of proof for Theorem 3 lends itself to generalizations, setting the stage for future work regarding more types of creature forcings, as well as possible density versions of Theorem 3 and variants in the vein of [17], in which Todorcevic and Tyros proved the density version of Theorem 4. In Section 4, we show that Examples 2.10, 2.11, and 2.13 in [15] have dense subsets forming topological Ramsey spaces. Theorem 3 is applied to prove the Axiom A.4 for Examples 2.10 and 2.11; the Hales-Jewett Theorem is used to prove the Axiom A.4 for Example 2.13.

The motivation for this line of investigation is several-fold. When a forcing has a dense set forming a topological Ramsey space, it makes available Ramsey-theoretic techniques aiding investigations of the properties of the generic extensions and the related generic ultrafilter. In particular, it makes investigations of forcing over \( L(\mathbb{R}) \) reasonable, as all subsets of the space in \( L(\mathbb{R}) \) are Ramsey. Further, by work of Di Prisco, Mijares, and Nieto in [4], in the presence of a supercompact cardinal, the generic ultrafilter forced by a topological Ramsey space, partially ordered by almost reduction, has complete combinatorics in over \( L(\mathbb{R}) \). Having at one’s disposal the Abstract Ellentuck Theorem or the Abstract Nash-Williams Theorem aids in proving canonical equivalence relations on fronts and barriers, in the vein of Pudlák and Rödl [12]. This in turn makes possible investigations of initial Rudin-Keisler and Tukey structures below these generic ultrafilters in the line of [13], [8], [9], [7], [5], and [6].

For the sake of space, we only include in Section 2 the basics of topological Ramsey spaces needed to understand the present work and refer the reader to Todorcevic’s book [16] for a more thorough background. Likewise, we do not attempt to adequately present background material on creature forcing. However, we include throughout this paper references to Roslanowski and Shelah’s book [14] and their paper [15] so that the interested reader can pursue further this line of research.

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On a personal note, I would like to thank Alan Dow for his inspiring and encouraging influence on my early and present mathematics. He and his work are truly exceptional. Happy Birthday, Alan!

2. Basics of topological Ramsey spaces

A brief review of topological Ramsey spaces is provided in this section for the reader’s convenience. Building on seminal work of Carlson and Simpson in [2],
Todorcevic distilled key properties of the Ellentuck space into four axioms, A.1 - A.4, which guarantee that a space is a topological Ramsey space. (For further background, the reader is referred to Chapter 5 of [16].) The axioms A.1 - A.4 are defined for triples (R, ≤, r) of objects with the following properties: R is a nonempty set, ≤ is a quasi-ordering on R, and r : R × ω → AR is a surjective map producing the sequence (r_n(·) = r(·, n)) of restriction maps, where AR is the collection of all finite approximations to members of R. For u ∈ AR and X, Y ∈ R,

(1) [u, X] = {Y ∈ R : Y ≤ X and (∃n) r_n(Y) = u}.

For each n < ω, AR_n = {r_n(X) : X ∈ R}.

A.1 (1) r_0(X) = ∅ for all X ∈ R.

(2) X ≠ Y implies r_n(X) ≠ r_n(Y) for some n.

(3) r_m(X) = r_n(Y) implies m = n and r_k(X) = r_k(Y) for all k < n.

According to A.1 (3) for each u ∈ AR there is exactly one n for which there exists an X ∈ R satisfying u = r_n(X). This n is called the length of u and we write |u| = n. We use the abbreviation [n, X] to denote [r_n(X), X]. For u, v ∈ AR, we write u ⊆ v if and only if (∃X ∈ R)(∀m ≤ n ∈ ω)(∀u = r_m(X) ∧ v = r_m(Y)). We write u ⊊ v if and only if u ⊆ v and u ≠ v.

A.2 There is a quasi-ordering ≤_fin on AR such that

(1) {v ∈ AR : v ≤_fin u} is finite for all u ∈ AR,

(2) Y ⊊ X iff (∀n)(∃m) r_n(Y) ≤_fin r_m(X),

(3) ∀u, v, y ∈ AR[y ⊊ v ∧ v ≤_fin u ⇒ ∃x ⊊ u (y ≤_fin x)].

The number depth_X(u) is the least n, if it exists, such that u ≤_fin r_n(X). If such an n does not exist, then we write depth_X(u) = ∞. If depth_X(u) = n < ∞, then [depth_X(u), X] denotes [r_n(X), X].

A.3 (1) If depth_X(u) < ∞ then [u, Y] ≠ ∅ for all Y ∈ [depth_X(u), X].

(2) Y ⊊ X and [u, Y] ≠ ∅ imply that there is Y' ∈ [depth_X(u), X] such that ∅ ≠ [u, Y'] ⊊ [u, Y].

Additionally, for n > |u|, let r_n[u, X] denote the collection {r_n(Y) : Y ∈ [u, X]}. A.4 If depth_X(u) < ∞ and if O ⊆ AR_{|u|+1}, then there is Y ∈ [depth_X(u), X] such that r_{|u|+1}[u, Y] ⊆ O or r_{|u|+1}[u, Y] ⊆ O^c.

The Ellentuck topology on R is the topology generated by the basic open sets [u, X]; it refines the metric topology on R, considered as a subspace of the Tychonoff cube AR^N. Given the Ellentuck topology on R, the notions of nowhere dense, and hence of meager are defined in the usual way. We say that a subset X of R has the property of Baire iff X = O ∩ M for some Ellentuck open set O ⊆ R and Ellentuck meager set M ⊆ R.

Definition 1 ([16]). A subset X of R is Ramsey if for every ∅ ≠ [u, X], there is a Y ∈ [u, X] such that [u, Y] ⊊ X or [u, Y] ∩ X = ∅. X ⊆ R is Ramsey null if for every ∅ ≠ [u, X], there is a Y ∈ [u, X] such that [u, Y] ∩ X = ∅.

A triple (R, ≤, r) is a topological Ramsey space if every subset of R with the property of Baire is Ramsey and if every meager subset of R is Ramsey null.

The following result can be found as Theorem 5.4 in [16].
Theorem 2 (Abstract Ellentuck Theorem). If \((R, \leq, r)\) is closed (as a subspace of \(\mathcal{AR}^{\mathbb{N}}\)) and satisfies axioms A.1, A.2, A.3, and A.4, then every subset of \(R\) with the property of Baire is Ramsey, and every meager subset is Ramsey null; in other words, the triple \((R, \leq, r)\) forms a topological Ramsey space.

3. A variant of the product tree Ramsey theorem

The main theorem of this section, Theorem 3, is a Ramsey theorem on unbounded finite products of finite sets. This is a variant of Theorem 4 below, with the strengthenings that exactly one of the entries \(K_i\) in each finite product is replaced with \([K_i]_k\) and \(l\) is allowed to vary over all numbers less than or equal to the length of the product, and the weakening that some of the chosen subsets may have cardinality one. It seems that a full strengthening of Theorem 4 of the form where the index of the \(k\)-sized subsets is allowed to vary over every index \(l\) may not be possible (see Remark 2). The conclusion of Theorem 3 is what is needed to prove Axiom A.4 for two of the examples of forcing with pure candidates in the next section; it is the essence of the pigeonhole principle for \(R_k\) on \([k-1, \bar{t}]\), for \(\bar{t}\) in a particular dense subset of the creature forcing. The hypothesis in Theorem 3 that the sizes of the \(K_j\) grow as \(j\) increases lends itself to our intended applications.

Throughout, for \(l \leq n\), \([K_l]_k \times \prod_{j \in (n+1) \setminus \{l\}} K_j\) is used to denote \(K_0 \times \cdots \times K_{l-1} \times [K_l]_k \times K_{l+1} \times \cdots \times K_n\).

**Theorem 3.** Given \(k \geq 1\), a sequence of positive integers \((m_0, m_1, \ldots)\), sets \(K_j, j < \omega\), such that \(|K_j| \geq j + 1\), and a coloring

\[
c : \bigcup_{n < \omega} \bigcup_{l \leq n} ([K_l]_k \times \prod_{j \in (n+1) \setminus \{l\}} K_j) \to 2,
\]

there are infinite sets \(L, N \subseteq \omega\) such that, enumerating \(L\) and \(N\) in increasing order, \(l_0 \leq n_0 < l_1 \leq n_1 < \ldots\), and there are subsets \(H_j \subseteq K_j, j < \omega\), such that \(|H_{i_j}| = m_i\) for each \(i < \omega\), \(|H_j| = 1\) for each \(j \in \omega \setminus L\), and \(c\) is constant on

\[
\bigcup_{n \in N} \bigcup_{l \in (n+1) \setminus \{l\}} ([H_l]_k \times \prod_{j \in (n+1) \setminus \{l\}} H_j).
\]

Theorem 3 is a variant of the following product tree Ramsey theorem, (Lemma 2.2 in [3] and Theorem 3.21 in [16]), which we now state since it will be used in the proof of Theorem 3. Let \(\mathbb{N}^+\) denote the set of positive integers.

**Theorem 4 (Di Prisco-Llopis-Todorcevic, [3]).** There is an \(R : [\mathbb{N}^+]^{<\omega} \to \mathbb{N}^+\) such that for every infinite sequence \((m_j)_{j < \omega}\) of positive integers and for every coloring

\[
c : \bigcup_{n < \omega} \prod_{j \leq n} R(m_0, \ldots, m_j) \to 2,
\]

there exist \(H_j \subseteq R(m_0, \ldots, m_j), |H_j| = m_j,\) for \(j < \omega\), such that \(c\) is constant on the product

\[
\prod_{j \leq n} H_j
\]

for infinitely many \(n < \omega\).
The proof of Theorem 3 closely follows the line of proof of Theorem 4 as presented in [16]. It will follow from Corollary 10 (proved via Lemmas 8 and 9 and Theorem 6) along with a final application of Theorem 4. The following lemma and its proof are minor modifications of Lemma 2.1 in [3] (see also Lemma 3.20 in [16]), the only difference being the use of $|H_0|^k$ in place of $H_0$. We make the notational convention that for $n = |H_0|^k \times \prod_{j=1}^{n} H_j$ denotes $|H_0|^k$.

**Lemma 5.** For any given $k \geq 1$ and sequence $(m_j)_{j<\omega}$ of positive integers, there are numbers $S_k(m_0, \ldots, m_j)$ such that for any $n < \omega$ and any coloring

$$c : [S_k(m_0)]^k \times \prod_{j=1}^{n} S_k(m_0, \ldots, m_j) \to 2,$$

there are sets $H_j \subseteq S_k(m_0, \ldots, m_j)$, $j \leq n$, such that $|H_j| = m_j$ and $c$ is monochromatic on $[H_0]^k \times \prod_{j=1}^{n} H_j$.

**Proof.** Let $S_k(m_0)$ be the least number $r$ such that $r \to (m_0)^2$. This satisfies the lemma when $n = 0$. Now suppose that $n \geq 1$ and the numbers $S_k(m_0, \ldots, m_j)$, $j < n$, have been obtained satisfying the lemma. Let $N$ denote the number $|S_k(m_0)|^k \cdot S_k(m_0, m_1) \cdots S_k(m_0, \ldots, m_{n-1})$, and let $S_k(m_0, \ldots, m_n) = m_n \cdot 2^N$. Given a coloring $c : [S_k(m_0)]^k \times \prod_{j=1}^{n} S_k(m_0, \ldots, m_j) \to 2$, for each $t \in [S_k(m_0)]^k \times \prod_{j=1}^{n} S_k(m_0, \ldots, m_j)$, let $c_t$ denote the coloring on $S_k(m_0, \ldots, m_n)$ given by $c_t(x) = c(t \cdot x)$, for $x \in S_k(m_0, \ldots, m_n)$. Let $(t_i : i < N)$ be an enumeration of the members of $[S_k(m_0)]^k \times \prod_{j=1}^{n-1} S_k(m_0, \ldots, m_j)$, and let $K_0 = S_k(m_0, \ldots, m_n)$. Given $i < N$ and $K_i$, take $K_{i+1} \subseteq K_i$ of cardinality $m_n \cdot 2^{N-(i+1)}$ such that $c_{t_i}$ is constant on $\{t_i \cdot x : x \in K_{i+1}\}$. By induction on $i < N$, we obtain $K_N \subseteq S_k(m_0, \ldots, m_n)$ of size $m_n$ such that for each $i < N$, $c_{t_i}$ is constant on $K_N$. Let $H_N = K_N$. Now let $c'$ be the coloring on $[S_k(m_0)]^k \times \prod_{j=1}^{n-1} S_k(m_0, \ldots, m_j)$ given by $c'(t) = c(t \cdot x)$, for any (every) $x \in H_N$. By the induction hypothesis, there are $H_j \subseteq S_k(m_0, \ldots, m_j)$ of cardinality $m_j$, $j < n$, such that $c'$ is constant on $[H_0]^k \times \prod_{j=1}^{n} H_j$. Then $c$ is constant on $[H_0]^k \times \prod_{j=1}^{n} H_j$. □

**Remark 1.** The case $k = 1$ is simply a re-statement of Lemma 2.1 in [3]. If $m_0 < k$, then the set $[S_k(m_0)]^k$ is the emptyset, so the whole product is empty and the lemma is vacuously true.

**Remark 2.** If one wants a generalization of Theorem 4 where the placement of the $k$-sized subsets can range over all $l$, it seems that only a finite version may be possible, as the bounds on the sizes of the sets needed to guarantee homogeneity depend both on $k$ and the number of products. The proof of the following statement proceeds very similarly to the proof of Lemma 5 with the difference that one must consider $n$ different products instead of just one.

Given $k \geq 1$ and $n < \omega$, there is a function $S_{k,n} : [\mathbb{N}^+]^{\leq n} \to \mathbb{N}^+$, depending on both $k$ and $n$, such that for each sequence $(m_j)_{j \leq n}$ of positive integers, for each coloring

$$c : \bigcup_{l \leq n} [S_{k,n}(m_0, \ldots, m_l)]^k \times \prod_{j \in (n+1) \setminus \{l\}} S_{k,n}(m_0, \ldots, m_j) \to 2,$$

there are subsets $H_j \subseteq S_{k,n}(m_0, \ldots, m_j)$ such that for each $l \leq n$, $|H_j| = m_j$ and $c$ is constant on $[H_l]^k \times \prod_{j \in (n+1) \setminus \{l\}} H_j$. 
As this theorem is not applied in this article and the proof takes up much room for notational reasons, we merely note here the first few such numbers. Let \( r^k_1(m) \) denote the least number \( r \) such that \( r \to (m)^2_2 \), and let \( r^{k+1}_1(m) \) denote the least number \( r \) such that \( r \to (r^k_1(m))^2_2 \). For \( n = 0 \), \( S_{k,0}(m_0) = r_k(m_0) \). For \( n = 1 \), the numbers \( S_{k,1}(m_0, m_1) = r_{S_{k,1}(m_0)}(m_1) \). 

The point is that a general statement like this for infinite sequences \((m_j)_{j<\omega}\) would a priori seem the natural route to proving Theorem 3, but as it only holds for finite sequences, we had to find a different means of proving the main theorem of this section.

The following generalizes Theorem 4, the first \( R(m_0) \) being replaced by \([R_k(m_0)]^k\), and provides a step toward the proof of Theorem 5. Its proof comes after Lemma 9.

**Theorem 6.** Given \( k \geq 1 \), there is a function \( R_k : [N^+]^\omega \to \mathbb{N}^+ \) such that for each sequence \((m_j)_{j<\omega}\) of positive integers, for each coloring

\[
c : \bigcup_{n<\omega} [R_k(m_0)]^k \times \prod_{j=1}^n R_k(m_0, \ldots, m_j) \to 2,
\]

there are subsets \( H_j \subseteq R_k(m_0, \ldots, m_j) \) such that \(|H_j| = m_j\) and \( c \) is constant on

\[
[H_0]^k \times \prod_{j=1}^n H_j
\]

for infinitely many \( n \).

The following Ramsey Uniformization Theorem, due to Todorcevic, appears (without proof) as Theorem 1.59 in [16] and is essential to Lemma 9 below. As previously no proof was available in the literature and at the request of the referee, the proof, as communicated to the author by Todorcevic, is included here, with notation slightly modified to cohere with this article. In the following theorem and proof, the projective hierarchy refers to the metric topology on the Baire space, \( [\omega]^\omega \).

**Theorem 7** (Ramsey Uniformization Theorem, [16]). Suppose \( X \) is a Polish space and \( R \) is a coanalytic subset of the product \([\omega]^\omega \times X\) with the property that for all \( M \subseteq [\omega]^\omega \) there is \( x \in X \) such that \( R(M, x) \) holds. Then there is an infinite subset \( M \) of \( \omega \) and a continuous map \( F : [M]^\omega \to X \) such that \( R(N, F(N)) \) holds for all \( N \subseteq [M]^\omega \).

**Proof.** Let \( R \) be as in the hypothesis. By the Kondō Uniformization Theorem [11], there is a coanalytic function \( f \subseteq R \) which uniformizes \( R \), meaning that \( R(N, f(N)) \) for each \( N \) in the projection of \( R \) to \([\omega]^\omega\). Since \( X \) is Polish, there is a countable base for its topology, say \( \{U_i : i < \omega\} \). Then for each \( i < \omega \), \( f^{-1}(U_i) \) is \( \Sigma^1_2 \), since \( f \) is \( \Pi^1_1 \). Since \( \Sigma^1_2 \) sets have the Ramsey property (with respect to the Ellentuck topology), the following fusion construction to obtain \( M \) will complete the proof.

Since \( f^{-1}(U_0) \) has the Ramsey property, there is an \( N_0 \subseteq [\omega]^\omega \) such that either \([N_0]^\omega \subseteq f^{-1}(U_0)\) or else \([N_0]^\omega \cap f^{-1}(U_0) = \emptyset\). Let \( m_0 = \min(N_0) \) and let \( N'_1 = \)
$N_0 \setminus \{m_0\}$. Suppose now $k \geq 1$ and we have chosen $N_0 \supseteq \ldots \supseteq N_{k-1}$, such that letting $m_i = \min(N_i)$, we have $m_0 < \ldots < m_{k-1}$ and for each $i < k$,

$$\forall s \in \{m_j : j < i\} \cup \{s, N_i\} \subseteq f^{-1}(U_i) \lor [s, N_i] \cap f^{-1}(U_i) = \emptyset.$$  
(2)

Let $N'_k$ denote $N_{k-1} \setminus \{m_{k-1}\}$ and enumerate $\mathcal{P}(\{m_j : j < k\})$ as $\langle t_l : l < 2^k \rangle$. Since $f^{-1}(U_k)$ is Ramsey, there is an $N'_k \in [t_0, N'_k]$ such that either $[t_0, N'_k] \subseteq f^{-1}(U_k)$ or else $[t_0, N'_k] \cap f^{-1}(U_k) = \emptyset$. For $l < 2^k - 1$, having chosen $N'_l$, take $N'_l+1 \in [t_{l+1}, N'_l]$ such that either $[t_{l+1}, N'_l+1] \subseteq f^{-1}(U_k)$ or else $[t_{l+1}, N'_l+1] \cap f^{-1}(U_k) = \emptyset$. At the end of these $2^k$ many steps, take $N_k$ to be $N'_{2^k-1}$, $m_k = \min(N_k)$, and set $N_{k+1} = N_k \setminus \{m_k\}$. Note that equation (2) now holds with $k$ substituted for $i$.

Let $M = \{m_k : k < \omega\}$. $M$ is infinite, since $m_k < m_{k+1}$, for all $k < \omega$. Let $F$ denote $f \upharpoonright [M]^\omega$. To show that $F$ is continuous, it suffices to show that for each $i < \omega$, $F^{-1}(U_i)$ is open in $[M]^\omega$. A standard and useful notation is to let $M/m_{i-1}$ denote the set of members of $M$ strictly greater than $m_{i-1}$. Let $m_{-1}$ denote $-1$ so that $M_0/m_{-1}$ equals $M$.

**Claim 8.** $F^{-1}(U_i) = \bigcup\{\{s, M/m_{i-1}\} : s \subseteq \{m_j : j < i\} \text{ and } [s, N_i] \subseteq f^{-1}(U_i)\}$.

**Proof.** First note that $\{s, M/m_{i-1}\}$ for all $s \subseteq \{m_j : j < i\}$ partitions $[M]^\omega$ into a disjoint union of finitely many clopen sets, in the subspace topology on $[M]^\omega$ inherited from $[\omega]^\omega$. For each $s \subseteq \{m_j : j < i\}$, by equation (2), one of two cases holds: If $[s, N_i] \subseteq f^{-1}(U_i)$, then

$$[s, M/m_{i-1}] = [s, N_i] \cap [M]^\omega \subseteq f^{-1}(U_i) \cap [M]^\omega = F^{-1}(U_i).$$  
(3)

If $[s, N_i] \cap f^{-1}(U_i) = \emptyset$, then since $[s, M/m_{i-1}] \subseteq [s, N_i]$ and $F^{-1}(U_i) \subseteq f^{-1}(U_i)$, it follows that $[s, M/m_{i-1}] \cap f^{-1}(U_i) = \emptyset$. Thus, the Claim holds.

Therefore, $F^{-1}(U_i)$ is a union of open sets in $[M]^\omega$; hence, $F$ is a continuous function from $[M]^\omega$ into $X$. □

Given $k \geq 1$ and $M \in [\omega]^\omega$, letting $\{m_j : j < \omega\}$ be the increasing enumeration of $M$, the notation $M_e \longleftrightarrow^k M_e$ means that for each 2-coloring $c : \bigcup_{n<\omega}([m_1]^k \times \prod_{j=1}^n m_{2j+1}) \to 2$, there are $H_j \subseteq m_{2j+1}$ such that $|H_j| = m_{2j}$ and $c$ is constant on $[H_j]^k \times \prod_{j=1}^n H_j$ for infinitely many $n$. The following lemma and its proof are almost identical with those of Lemma 3.18 in [16], the only changes being the substitution of $\bigcup_{n<\omega}([\omega]^k \times \omega^{n-1})$ for the domain of the function $c$ in place of $\omega^{<\omega}$, the substitution of $[\omega]^k$ for one of the copies of $\omega$, and an application of Lemma 5 in place of the application of Lemma 3.20 in [16]. Thus, we omit its proof.

**Lemma 9.** For each $k \geq 1$, there is an infinite subset $N \subseteq \omega$ such that $M_o \longleftrightarrow^k M_e$ for each $M \in [N]^\omega$.

The next proof proceeds by slight modification to the proof of Theorem 4 replacing $R(m_0)$ there with $[R_k(m_0)]^k$ and replacing an instance of Lemma 3.18 in [16] with Lemma 5.

**Proof of Theorem 4.** Pick an infinite subset $N = \langle n_p \rangle_{p < \omega}$ of positive integers enumerated in increasing order and satisfying Lemma 5. For each $j < \omega$, set

$$R_k(m_0, \ldots, m_j) = n_{2^\left(\sum_{i=0}^j m_i\right)+1}.$$
Then for every infinite sequence \((m_j)_{j<\omega}\) of positive integers, if we let
\[
P = \{n \in \mathbb{N} : j \in \omega, \varepsilon < 2\},
\]
then \(P\) is an infinite subset of \(N\) satisfying \(P = (R_k(m_0, \ldots, m_j))_{j<\omega}\), while the sequence \(P_x\) pointwise dominates our given sequence \((m_j)_{j<\omega}\). By our choice of \(N\), it follows that \(P_o \longrightarrow^k P_e\). \(P_o\) supplies the infinitely many levels of \(n\) satisfying the theorem.

The following corollary forms the basis of the proof of Theorem 3 below.

**Corollary 10.** Let \(L, N\) be infinite subsets of \(\omega\) such that \(l_0 \leq n_0 < l_1 \leq n_1 < \ldots\). Let \(k \geq 1, m_0 \geq 1,\) and \(K_j, j \geq l_0,\) be nonempty sets with \(|K_{l_0}| = R_k(m_0), |K_j| \geq i\) for each \(i \geq 1,\) and \(|K_j| = 1\) for each \(j \in (l_0, \omega) \setminus L\). Then for each infinite sequence \((\mathbb{N}, \mathbb{N})\) of positive integers, if we let
\[
\mathcal{H}_j \subseteq \{K_{l_0}, \ldots, K_{l_1}, \ldots\},
\]
we have \(|\mathcal{H}_j| = \#(\mathbb{N})\).

**Proof.** Let \(r < \omega\) be fixed. Take \((p, p) \subseteq \omega\) a strictly increasing sequence so that \(i_0 = 0\) and \(|K_{l_0}| \geq R_k(m_0, r + 1, \ldots, r + p)\). For each \(j \in (l_0, \omega) \setminus \{l_p : p \geq 1\},\) take \(H_j \subseteq K_j\) of size one. Then the coloring \(c\) on
\[
\bigcup_{n \in \mathbb{N}} \left(\mathcal{H}_0\right)^k \times \prod_{j \in (l_0, n]} K_j
\]
induces a coloring \(c'\) on \(\bigcup_{i \subseteq \omega} [\mathcal{H}_0]^k \times \prod_{j \in [i]} J_q,\) where \(J_q = K_{l_1}\), as follows: For \(p < \omega\) and \((X_0, x_1, \ldots, x_p) \subseteq [\mathcal{H}_0]^k \times \prod_{j \in [p]} J_q,\) letting \(Y_0 = X_0, y_{l_1} = x_2,\) and for each \(j \in (l_0, n_p) \setminus \{l_i : q \leq p\}\) letting \(y_j\) denote the member of \(H_j\), we define \(c'(X_0, x_1, \ldots, x_p) = c(Y_0, y_{l_0} + 1, \ldots, y_{n_p})\). Apply Theorem 6 to \(c'\) to obtain \(H_{l_0} \in \mathbb{N}^m_s\), subsets \(H_{l_p} \subseteq [K_{l_0}]^{l_0 + p}\) for each \(p \geq 1\), and an infinite set \(S\) such that \(c'\) is constant on \(\bigcup_{p \in [p]} [\mathcal{H}_0]^k \times \prod_{q \leq p} H_{l_p}\). Then letting \(N' = \{n_p : p \in S\} \subseteq \mathbb{N},\) \(c\) is constant on \(\bigcup_{n \in N'} [\mathcal{H}_0]^k \times \prod_{i < l_1 \leq n} H_{l_i}^0\). Letting \(L' = \{l_p : p \in S\},\) finishes the proof.

Now we are equipped to prove Theorem 3.

**Proof of Theorem 3.** Take \(l_0\) least such that \(|K_{l_0}| \geq R_k(R(m_0))\), and let \(L_0 = [l_0, \omega)\). For each \(j < l_0\), take some \(H_j \subseteq [K_{l_0}]^j\) and let \(h \downarrow l_0\) denote \(\prod_{j < l_0} H_j\). Then \(h\) restricted to \(\bigcup_{n \in N_0} \left(\mathcal{H}_0\right)^k \times \prod_{j \in [l_0, n]} K_j\) induces a 2-coloring on \(\bigcup_{n \in N_0} [K_{l_0}]^k \times \prod_{j \in (l_0, n]} K_j\). By Corollary 10, there are infinite \(L' \subseteq L_0\) and \(N' \subseteq N_0\) such that \(l_0 = l' = l_0 < l' \leq l_1 < \ldots\), and there are subsets \(H_{l_0}^0 \subseteq K_{l_0}\), \(j \geq l_0,\) such that \(|H_{l_0}^0| = R(m_0), |H_j^0| = i\) for each \(i \geq 1, |H_j^0| = 1\) for each \(j \in \omega \setminus L_0\), and \(c\) is constant on \(\bigcup_{n \in N'} (h \downarrow l_0) \times [H_{l_0}^0]^k \times \prod_{j \in (l_0, n]} H_{l_0}^0\).

Let \(H_{l_0} = H_{l_0}^0\), and let \(n_0 = \min(N')\). Then \(n_0 \geq l_0\). Let \(R_k^0(m)\) denote \(R_k(m)\) and in general, let \(R_k^{l_1+1}(m)\) denote \(R_k(R_k^0(m))\). Fix an \(l_1 \in L_0\) such that \(l_1 > n_0\) and \(|H_{l_1}^0| \geq R_k^0(R(m_0, m_1))\). For \(j \in (l_0, l_1)\), fix some \(H_j \in [H_j^0]\), and
let $h \upharpoonright l_1$ denote $\prod_{i \in l_1 \setminus \{l_0\}} H_j$. Enumerate $H_{l_0}$ as $\{h_{l_0}^i : i < m_0\}$. Successively apply Corollary $\Box$ $R(m_0)$ times to obtain $L_1 \subseteq L_0'$ with $\min(L_1) = l_1$. $N_1 \subseteq N_0'$, $H_{l_1} \subseteq K_{l_1}$ of cardinality $R(m_0, m_1)$, and subsets $H_j^1 \subseteq K_j$ for $j \in [l_1, \omega)$, such that listing $L_1$ as $l_1 = l_1^1 < l_1^2 < \ldots$ we have $|H_j^1| \geq 1$ and satisfying the following: For each fixed $h_{l_0}^i \in H_{l_0}$, the coloring $c$ is constant on

$$\bigcup_{n \in N_1} (h \upharpoonright l_1) \times \{h_{l_0}^i\} \times [H_l^1]^k \times \prod_{j \in (l_1, n]} H_j^1.$$ 

In general, suppose for $p \geq 1$, we have fixed $l_0 = n_0 < \cdots < l_p \leq n_p$, and chosen infinite sets $L_p, N_p$ with $l_p = \min(L_p)$ and $l_p = p_0^p \leq n_0^p < p^p_{p+1} \leq n^p_{p+1} < \cdots$ and sets $H_j \subseteq K_j$ for $j \in [l_p, \omega)$ such that the following hold:

(1) for each $i \leq p$, $|H_{l_i}| = R(m_0, \ldots, m_i)$,
(2) for each $l \in l_p \setminus \{l_i : i < p\}$, $|H_l| = 1$,
(3) for each $i > p$, $|H^p_{l_i}| \geq i$,
(4) and for each $j \in (l_p, \omega) \setminus L_p$, $|H^p_j| = 1$.

Let $h \upharpoonright l_p$ denote $\prod_{j \in l_p \setminus \{l_0, \ldots, l_{p-1}\}} H_j$, which is a product of singletons. By our construction so far, we have ensured that for each sequence $\bar{x} \in \Pi_{i \leq p} H_{l_i}$, $c$ is constant on

$$\bigcup_{n \in N_p} h \upharpoonright l_p \times \bar{x} \times [H^p_{l_i}]^k \times \prod_{j \in (l_p, n]} H^p_j.$$ 

Let $n(p) = |\prod_{i \leq p} H_{l_i}|$. Fix $n_p \in N_p$ such that $n_p \geq l_p$ and take $l_{p+1} \in L_p$ such that $l_{p+1} > n_p$ and $|H^p_{l_{p+1}}| = R^{n(p)}(R(m_0, \ldots, m_{p+1}))$. After $n(p)$ successive applications of Corollary $\Box$ we obtain $L_{p+1} \subseteq L_p$ and $N_{p+1} \subseteq N_p$ with $\min(L_{p+1}) = l_{p+1}$, subsets $H_j \subseteq K_j$ for $j \in (l_p, l_{p+1}]$ and sets $H^p_{l_{p+1}} \subseteq H^p_j$ for $j > l_{p+1}$ such that the following hold:

(1) $|H^p_{l_{p+1}}| = R(m_0, \ldots, m_{p+1})$,
(2) for each $l \in (l_p, l_{p+1})$, $|H_l| = 1$,
(3) for each $i > p$, $|H^p_{l_i}| \geq i$,
(4) and for each $j \in (l_{p+1}, \omega) \setminus L_{p+1}$, $|H^p_j| = 1$;

and moreover, letting $h \upharpoonright l_{p+1} = \prod_{j \in l_{p+1} \setminus \{l_0, \ldots, l_p\}} H_j$, for each $\bar{x} \in \Pi_{i \leq p} H_{l_i}$, $c$ is constant on

$$\bigcup_{n \in N_{p+1}} (h \upharpoonright l_{p+1}) \times \bar{x} \times [H^p_{l_{p+1}}]^k \times \prod_{j \in (l_{p+1}, n]} H^p_{l_{p+1}}.$$ 

Then fix an $n_{p+1} \in N_{p+1}$ such that $n_{p+1} \geq l_{p+1}$.

In this manner, we obtain $L = \{l_i : i < \omega\}$ and $N = \{n_i : i < \omega\}$ such that $l_0 \leq n_0 < l_1 \leq n_1 < l_2 \leq n_2 < \ldots$, and $H_j \subseteq K_j$, $j < \omega$, such that $|H_{l_i}| = R(m_0, \ldots, m_i)$ for each $i < \omega$, $|H_j| = 1$ for each $j \in \omega \setminus L$, and for each $p < \omega$, for each $\bar{x} \in \Pi_{i \leq p} H_{l_i}$, $c$ is constant on

$$\bigcup_{n \in N \cap (l_p, \omega]} (h \upharpoonright l_p) \times \bar{x} \times [H^p_{l_p}]^k \times \prod_{j \in (l_p, n]} H_j,$$

Defining $c'(\bar{x})$ to be this constant color induces a 2-coloring on $\bigcup_{n \in \omega} \prod_{i \leq n} H_{l_i}$. Since each $|H_{l_i}| = R(m_0, \ldots, m_i)$, we may apply Theorem $\Box$ to obtain $H^*_i \subseteq H_{l_i}$ of cardinality $m_i$ and an infinite subset $N^* \subseteq N$ such that $\bar{c}$ is constant on $\bigcup_{n \in N^*} \prod_{i \leq n} H^*_i$.
Then letting $H_j^* = H_j$ for $j \notin L$, and letting $L^*$ be any subset of $\{l_i : i < \omega\}$ such that $l_0^* \leq n_0^* < l_1^* \leq n_1^* < \ldots$, $c$ is constant on

$$\bigcup_{n \in \mathbb{N}^*} \bigcup_{l \in L^* \cap (n+1)} [H_l^*]^k \times \prod_{j \in (n+1) \setminus \{l\}} H_j^*.$$ 

\[ \square \]

4. Topological Ramsey spaces as dense subsets in three examples of creature forcings

In [15], Rosłanowski and Shelah proved partition theorems on countable sets of finitary functions denoted $F$ (see Definition 11 below). Their proofs involved using the subcomposition operation $\Sigma$ to define a binary relation giving rise to a semigroup, and then proving the existence of an idempotent ultrafilter (or a sequence of idempotent ultrafilters in the tight case) by utilizing the Glazer technique including applications of Ellis’ Lemma. These partition theorems, stated as Observation 2.8 (3) and Conclusions 3.10 and 4.8 in [15], show that, under certain assumptions on the creating pair, given any finite partition of $F$, there is a pure candidate such that the collection of possibilities it codes is contained in one piece of the partition. In this section, we show that for three of these examples, the collections of pure candidates contain dense subsets which form topological Ramsey spaces. We obtain as corollaries Conclusion 4.8 for Example 2.11 (see Proposition 11 below) and Observation 2.8 (3) and Conclusions 3.10 and 4.8 in [15], show that, under certain assumptions on the creating pair, given any finite partition of $F$, there is a pure candidate such that the collection of possibilities it codes is contained in one piece of the partition. In each of the examples below, given a creating pair $(K, \Sigma)$, we shall form a dense subset of the pure candidates, call it $R(K, \Sigma)$, partially ordered by the partial ordering inherited from the collection of all pure candidates (see Definition 2.3 (2) in [15]).

To show that a dense subset of a collection of pure candidates $t$ forms a topological Ramsey space, it suffices by the Abstract Ellentuck Theorem 2 to define a notion of $k$-th approximation of $t$ and a quasi-ordering $\leq_{\text{fin}}$ on the collection of finite approximations (in our cases this will be a partial ordering), and then prove that the Axioms A.1 - A.4 hold. In each of the examples below, given a creating pair $(K, \Sigma)$, we shall form a dense subset of the pure candidates, call it $R(K, \Sigma)$, partially ordered by the partial ordering inherited from the collection of all pure candidates (see Definition 2.3 (2) in [15]).

For each $t = (t_0, t_1, \ldots) \in R(K, \Sigma)$, for $k < \omega$, we let $r_k(t) = (t_i : i < k)$. Thus, $r_0(t)$ is the empty sequence, and $r_1(t) = (t_0)$, a sequence of length one containing exactly one member of $K$. Let $\mathcal{AR}_k$ denote $\{r_k(t) : t \in R(K, \Sigma)\}$ and $\mathcal{AR}$ denote $\bigcup_{k < \omega} \mathcal{AR}_k$. For $a \in \mathcal{AR}$ and $t \in R(K, \Sigma)$, write $a \sqsubset t$ if and only if $a = r_j(t)$ for some $k < \omega$. The basic open sets in the Ellentuck topology are defined as $[a, t] = \{s \in R(K, \Sigma) : a \sqsubseteq s \text{ and } s \subseteq t\}$, for $a \in \mathcal{AR}$ and $t \in R(K, \Sigma)$. Define the partial ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ as follows: For $a, b \in \mathcal{AR}$, $b \leq_{\text{fin}} a$ if and only if there are $\bar{s}, \bar{t} \in R(K, \Sigma)$ and $j, k < \omega$ such that $t \leq \bar{s}$, $b = r_k(\bar{t})$, $a = r_j(\bar{s})$ and $m_{\text{up}}^{j-1} = m_{\text{up}}^{k-1}$. Abusing notation, we also write $a \leq_{\text{fin}} t$ if for some $k < \omega$, $a \leq_{\text{fin}} r_k(\bar{t})$. Let $\mathcal{AR}_k|t$ denote the set of all $a \in \mathcal{AR}_k$ such that $a \leq_{\text{fin}} t$, and let $\mathcal{AR}|t$ denote $\bigcup_{k < \omega} \mathcal{AR}_k|t$. 


Given this set-up, it is clear that A.1 holds. Since for each \((s_0, \ldots, s_{k-1}) \in AR\), \(\Sigma(s_0, \ldots, s_{k-1})\) is finite, A.2 (1) holds. A.2 (2) is simply the definition of the partial ordering \(\leq\) when restricted to \(R(K, \Sigma)\), and A.2 (3) is straightforward to check, using the definition of \(\leq_{fin}\). A.3 follows from the definition of \(\Sigma\). Thus, showing that the pigeonhole principle A.4 holds for these examples is the main focus of this section.

We now include some of the relevant creature forcing terminology. Knowing this vocabulary is not necessary for the proofs, but it is included here so the interested reader can make connections between the proofs here and the more general genre of creature forcings. In the following three examples, FP stands for forgetful partial, which is made explicit in Context 2.1 and Definition 2.2 in [15], and is reproduced here.

**Definition 11** ([15], page 356). Let \(H\) be a fixed function defined on \(\omega\) such that \(H(i)\) is a finite non-empty set for each \(i < \omega\). The set of all finite non-empty functions \(f\) such that \(\text{dom}(f) \subseteq \omega\) and \(f(i) \in H(i)\) (for all \(i \in \text{dom}(f)\)) will be denoted by \(F_H\).

An FP creature for \(H\) is a tuple
\[
t = (\text{nor}[t], \text{val}[t], \text{dis}[t], m^t_{dn}, m^t_{up})
\]
such that
- \(\text{nor}\) is a non-negative real number, \(\text{dis}\) is an arbitrary object, and \(m^t_{dn} < m^t_{up} < \omega\), and
- \(\text{val}\) is a non-empty finite subset of \(F_H\) such that \(\text{dom}(f) \subseteq [m^t_{dn}, m^t_{up})\) for all \(f \in \text{val}\).

Partial refers to the fact that the domains of the functions \(f \in \text{val}\) are allowed to be subsets of \([m^t_{dn}, m^t_{up})\), rather than the whole interval. For the definition of forgetful, see Definition 1.2.5 in [14]. For the definitions of an FFCC pair, \(\Sigma\), loose, tight, pure candidate, the set of possibilities \(\text{pos}(t)\) and \(\text{pos}^{tt}(t)\) on a pure candidate \(t\), the reader is referred to Definitions 2.2 and 2.3 in [15].

**Example 2.10 in [15]**. Let \(H_1(n) = n + 1\) for \(n < \omega\) and let \(K_1\) consist of all FP creatures \(t\) for \(H_1\) such that
- \(\text{dis}[t] = (u, i, A) = (u^t, i^t, A^t)\), where \(u \subseteq [m^t_{dn}, m^t_{up}), i \in u, \emptyset \neq A \subseteq H_1(i)\),
- \(\text{nor}[t] = \log_2(|A|)\),
- \(\text{val}[t] \subseteq \prod_{j \in u} H_1(j)\) is such that \(\{f(i) : f \in \text{val}[t]\} = A\).

For \(t_0, \ldots, t_n \in K_1\) with \(m^t_{up} = m^{t+1}_{dn}\), let \(\Sigma^t_1(t_0, \ldots, t_n)\) consist of all creatures \(t \in K_1\) such that
\[
m^t_{dn} = m^{t_0}_{dn}, \ m^t_{up} = m^{t_n}_{up}, \ u^t = \bigcup_{i \leq n} u^{t_i}, \ i^t = i^{t_i}, \ A^t \subseteq A^{t_i} \text{ for some } l^t \leq n,
\]
and \(\text{val}[t] \subseteq \{f_0 \cup \cdots \cup f_n : (f_0, \ldots, f_n) \in \text{val}[t_0] \times \cdots \times \text{val}[t_n]\}\). The collection of tight pure candidates \(PC^{tt}_1(K_1, \Sigma^t_1)\) is defined in Definition 2.3 in [15], pure meaning that \(\tilde{t}\) is an infinite sequence without a trunk. The partial ordering \(\leq\) on \(PC^{tt}_1(K_1, \Sigma^t_1)\) is defined by \(\tilde{t} \leq \hat{s}\) if and only if there is a strictly increasing sequence \((j_n)_{n<\omega}\) such that each \(t_n \in \Sigma_1(s_{j_n}, \ldots, s_{j_{n+1}} - 1)\).

**Remark 3**. Here, \(\tilde{t}\) is the stronger condition, as this reversal of the partial order notation of Rosłanowski and Shelah is better suited to the topological Ramsey space framework.
Rosłanowski and Shelah proved that \((K_1, \Sigma^*_1)\) is a tight FFCC pair (see Definition 2.2 in [15] with bigness (Definition 2.6, [15]) and \(t\)-multiadditivity (Definition 2.5, [15]), and is gluing (basically meaning that neighboring creatures can be glued together to obtain another creature - see Definition 2.1.7, [14]) on every \(\bar{t} \in \text{PC}_\infty^t(K_1, \Sigma^*_1)\). FFCC stands for smooth ([14, 1.2.5]) Forgetful monotonic ([14, 5.2.3]) strongly Finitary ([14, 1.1.3, 3.3.4]) Creature Creating pair.

Without going into more terminology than is necessary, we point out that in this particular example, for \(\bar{t} \in \text{PC}_\infty^t(K_1, \Sigma^*_1)\), the set of possibilities on the pure candidate \(\bar{t}\) is

\[
\text{pos}^t(\bar{t}) = \bigcup \{f_0 \cup \cdots \cup f_n : n \in \omega \land \forall i \leq n \ (f_i \in \text{val}[t_i])\}.
\]

For \(\bar{t} \in \text{PC}_\infty^t(K_1, \Sigma^*_1)\) and \(n < \omega, \bar{t} \upharpoonright n\) denotes \((t_n, t_{n+1}, \ldots)\), the tail of \(\bar{t}\) starting at \(t_n\). The following is Conclusion 4.8 in [15] applied to this example.

**Theorem 12** (Rosłanowski/Shelah, [15]). Let \(\bar{t} \in \text{PC}_\infty^t(K_1, \Sigma^*_1)\), and for each \(k < \omega\), let \(l_k \geq 1\) and \(d_k : \text{pos}^t(\bar{t} \upharpoonright k) \to l_k\) be given.

(a) There is an \(\bar{s} \leq \bar{t}\) in \(\text{PC}_\infty^t(K_1, \Sigma^*_1)\) such that \(m_{dn}^0 = m_{dn}^0\) and for each \(i < \omega\), if \(k\) is such that \(s_i \in \Sigma^t(\bar{t} \upharpoonright k)\), then \(d_k \upharpoonright \text{pos}^t(\bar{s} \upharpoonright i)\) is constant.

(b) If there is a fixed \(l \geq 1\) such that for each \(k < \omega\), \(l_k = l\), then there is an \(\bar{s} \leq \bar{t}\) in \(\text{PC}_\infty^t(K_1, \Sigma^*_1)\) and an \(l' < l\) for each \(i < \omega\), if \(k\) is such that \(s_i \in \Sigma^t(\bar{t} \upharpoonright k)\) and \(f \in \text{pos}^t(\bar{s} \upharpoonright i)\), then \(d_k(f) = l'\).

**Definition 13** (The space \(\mathcal{R}(\text{PC}_\infty^t(K_1, \Sigma^*_1), \preceq, r)\)). Let \(\mathcal{R}(\text{PC}_\infty^t(K_1, \Sigma^*_1))\) consist of those members \(\bar{t} \in \text{PC}_\infty^t(K_1, \Sigma^*_1)\) such that for each \(l < \omega\),

1. \(|A^t| = l + 1\), and
2. for each \(a \in A^t\), there is exactly one function \(g^t_a \in \text{val}[t_i]\) such that \(g^t_a(i^t) = a\).

It follows that for each \(\bar{t} \in \mathcal{R}(\text{PC}_\infty^t(K_1, \Sigma^*_1)), \) for each \(l < \omega, \text{val}[t_i] = \{g^t_a : a \in A^t\}\), and hence \(|\text{val}[t_i]| = |A^t| = l + 1\). We point out that \(r_1[0, \bar{t}]\) consists of those \(s \in \bigcup_{n<\omega} \Sigma^t(\bar{t} \upharpoonright n)\) such that \(|A^s| = |\text{val}[s]| = 1\).

**Theorem 14.** \(\mathcal{R}(\text{PC}_\infty^t(K_1, \Sigma^*_1), \preceq, r)\) is a topological Ramsey space which is dense in the partial ordering of all tight pure candidates \(\text{PC}_\infty^t(K_1, \Sigma^*_1)\).

**Proof.** Abbreviate \(\mathcal{R}(\text{PC}_\infty^t(K_1, \Sigma^*_1))\) as \(\mathcal{R}^t(K_1, \Sigma^*_1)\). The space \(\mathcal{R}^t(K_1, \Sigma^*_1)\) is clearly dense in \(\text{PC}_\infty^t(K_1, \Sigma^*_1)\). First we show A.4 holds for \(r_k[k-1, \bar{t}]\) for all \(k \geq 2\), and for \(r_1[0, \bar{t}] \cap \Sigma^t(\bar{t})\). Then we will use a fusion argument to obtain A.4 for \(r_1[0, \bar{t}]\).

**Claim 15.** Let \(\bar{t} \in \mathcal{R}^t(K_1, \Sigma^*_1)\), and let \(C_k\) denote \(r_1[0, \bar{t}] \cap \Sigma^t(\bar{t})\) if \(k = 1\), and \(r_k[k-1, \bar{t}]\) if \(k \geq 2\). Let \(c : C_k \to 2\) be given. Then there is an \(\bar{s} \leq \bar{t}\) in \(\mathcal{R}^t(K_1, \Sigma^*_1)\) with \(m_{dn}^0 = m_{dn}^0\) such that, if \(k = 1\), \(c\) is constant on \(r_1[0, \bar{s}] \cap \Sigma^t(\bar{s})\); and if \(k \geq 2\), then \(s \in [k-1, \bar{t}]\) and \(c\) is constant on \(r_k[k-1, \bar{s}]\).

**Proof.** Let \(k \geq 1\) be fixed. Each \(\bar{x} \in C_k\) is of the form \(\bar{x} = (t_0, \ldots, t_{k-2}, x_{k-1})\), where for some \(k-1 \leq l \leq n, x_{k-1} \in \Sigma^t_1(t_{k-1}, \ldots, t_n), i^{t_{k-1}} = i^{t_1}\) and \(A^{t_{k-1}} \in |A^t|^k\). For \(k = 1\), \(\bar{x}\) is simply \((x_0)\). Notice that \(x_{k-1}\) is completely determined by the sequence \((n, l, A^{t_{k-1}}, \{a_j : j \in [k-1, n] \setminus \{l\}\})\), where for each \(j \in [k-1, n] \setminus \{l\}, \), \(a_j\) is the member of \(A^t\) such that every member \(f \in \text{val}[x_{k-1}]\) satisfies \(f(i^t) = a_j\).
Therefore, \( c \) induces a coloring on
\[
\bigcup_{n \geq k-1} \left( \bigcup_{k-1 \leq l \leq n} [A^{|k|}]^k \times \prod_{j \in [k-1,n]\setminus \{l\}} A^{|\nu|} \right).
\]

Apply Theorem 3 to the sequence of sets \( A^{|j|}, j \geq k-1 \) to obtain infinite sets \( L, N \) and subsets \( H_j \subseteq A^{|j|} \) such that \( k-1 \leq l_0 \leq n_0 < l_1 \leq n_1 < \ldots \), and for each \( p < \omega, |H_p| = k + p \), and for each \( j \in \omega \setminus L \), \( |H_j| = 1 \); and moreover, \( c \) is constant on
\[
\bigcup_{n \in N} \bigcup_{l \in L \cap (n+1)} \Big[ |H|^k \times \prod_{|j| \in [k-1,n]\setminus \{l\}} H_j. \Big]
\]

Let \( \bar{s} \in \mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1) \) be defined as follows: \((s_0, \ldots, s_{k-2}) = r_{k-1}(\bar{t})\). For \( p \geq k-1 \), letting \( q = p - (k-1) \) and \( n_{k-1} = k-2 \), let \( s_p \) be the member of \( \Sigma^*_1(t_{n_{k-1}+1}, \ldots, t_{n_{p}}) \) such that \( \bar{t}^{s_p} = i^{s_q}, A^{t_p} = H_{t_q} \), and for all \( f \in \text{val}[s_p], f(i^{s_q}) \in H_j \) for each \( j \in (n_{q-1}, n_{q}] \). Then \( \bar{s} \) is a member of \( \mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1) \) with \( \bar{s} \leq \bar{t} \) satisfying Claim 15.

Thus, we have proved A.4 for \( r_{k}[k-1, \bar{t}] \), for all \( k \geq 2 \). As an step to proving A.4 for \( r_1[0, \bar{t}] \), we first prove the following Claim.

Claim 16. Given \( \bar{t} \in \mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1) \) and colorings \( c_k : \mathcal{A}\mathcal{R}_n | \bar{t} | \to l_k \), for some \( l_k \geq 1 \), there is an \( \bar{s} \leq \bar{t} \in \mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1) \) such that \( m^{|0|}_{dn} = m^{|0|}_{dn}, c_0 \) is constant on \( r_1[0, \bar{s}] \cap \Sigma^*_1(\bar{s}) \), and for each \( k \geq 1 \), the coloring \( c_k \) on \( r_{k+1}[k, \bar{s}] \) is constant.

Proof. This is a standard fusion argument. By Claim 15 there is an \( s^0 = (s^0_0, s^0_1, \ldots) \) in \( \mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1) \) with \( s^0 \leq \bar{t} \) such that \( m^{|0|}_{dn} = m^{|0|}_{dn} \) and \( c_0 \) is constant on \( r_1[0, \bar{s}] \cap \Sigma^*_1(s^0) \). Let \( s_0 = s^0_0 \). Suppose \( k \geq 1 \) and \( s^{k-1} \) has been chosen. Considering the coloring \( c_k \) restricted to \( r_{k+1}[k, s^{k-1}] \), Claim 15 implies there is an \( \bar{s}^k \in (k, s^{k-1}] \) for which \( c_k \) is constant on \( r_{k+1}[k, \bar{s}^k] \). Set \( s_k = \bar{s}^k \). Then \( \bar{s} = (s_0, s_1, \ldots) \leq \bar{t} \) is in \( \mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1) \) and satisfies the Claim.

Finally, to prove A.4 for \( \mathcal{A}\mathcal{R}_1 \), let \( c : r_1[0, \bar{t}] \to 2 \) be given. For each \( k < \omega \), define a coloring \( c_k : \mathcal{A}\mathcal{R}_k[\bar{t}] \to 2 \) by \( c_k(x_0, \ldots, x_k) = c(\text{min}(x_k)) \), where we let \( \text{min}(x) \) denote the member of \( \Sigma^*_1(\bar{t}) \) such that \( A^{\text{min}(x)} = \{ A^{|\nu|} \} \) and \( \text{val}[\text{min}(x)] \) is the singleton \( \{ f \} \) where \( f \) satisfies \( f(i^{s_q}) = \text{min}(A^{s_q}) \). Take \( \bar{s} \leq \bar{t} \) satisfying Claim 16. Then there is a strictly increasing sequence \( (k_j)_{j<\omega} \) such that each \( k_j \geq 2j + 1 \) and the coloring \( c_{k_j} \) is the same on \( r_{k_j+1}[k_j, \bar{s}] \), for all \( j < \omega \). Let \( \bar{v} \leq \bar{s} \) be the member of \( \mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1) \) such that for each \( j < \omega, v_j \in \Sigma^*_1(s_{k_j}, \ldots, s_{k_{j+1}-1}) \), \( i^{v_j} = i^{s_{k_{j+1}-1}} \), and \( A^{v_j} \) consists of the least \( j \) members of \( A^{s_{k_{j+1}-1}} \). Then \( \bar{v} \leq \bar{s} \) and \( c \) is constant on \( r_1[0, \bar{v}] \).

Thus, A.4 holds, and therefore, by the Abstract Ellentuck Theorem 2 and earlier remarks, \( (\mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1), \leq, r) \) is a topological Ramsey space.

We now show that Theorem 12 is recovered from Theorem 14. Let \( \bar{t}, l_k \) and \( d_k \) be as in the assumption of Theorem 12. Since \( \mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1) \) is dense in \( \text{PC}_\infty^\mathfrak{s}(K_1, \Sigma^*_1) \), we may without loss of generality assume \( \bar{t} \in \mathcal{R}^\mathfrak{s}(K_1, \Sigma^*_1) \). Then note that for each \( k < \omega, \text{pos}^\mathfrak{s}(\bar{t}, k) = \bigcup_{\text{val}[s_k]} : (s_0, \ldots, s_k) \in r_{k+1}[k, \bar{t}] \} \).
constant on $r_{k+1}[k,s]$. Let $\bar{w} \leq \bar{s}$ be the member of $R^\text{tt}(K_1, \Sigma^*_1)$ determined by $w_0 = s_0$, and for $n \geq 1$, $w_n$ is the member of $\Sigma^*_1(s_{2n-1},s_{2n})$ such that $i^{w_n} = i^{s_{2n}}$, $A^{w_n}$ is the set of the $n + 1$-least members of $A^{s_{2n}}$. (There is exactly one such member of $R^\text{tt}(K_1, \Sigma^*_1)$ with these properties.) Then $\bar{w} \leq \bar{s}$ with $m_{\text{dn}}^{w_0} = m_{\text{dn}}^{s_0}$ and for each $i < \omega$, for the $k$ such that $w_i \in \Sigma^*(t \mid k)$, $d_k$ is constant on $\text{pos}^\text{tt}(\bar{w} \mid t)$.

Part (b) of Theorem 12 follows immediately from A.4: Recalling that for each $s \in \mathcal{AR}_1$, $|\text{val}[s]| = 1$, define $c : \mathcal{AR}_1[t] \to l$ by $c(s) = d_k(s)(f^*)$, where $f^*$ is the member of $\text{val}[s]$ and $k(s)$ is the integer such that $s \in \Sigma^*_1(t \mid k(s))$. By A.4, there is an $\bar{s} \leq \bar{t}$ in $R^\text{tt}(K_1, \Sigma^*_1)$ such that $c$ is constant on $r_1[0, \bar{s}]$, and hence, $\bar{s}$ satisfies (b) of Theorem 12.

Remark 4. Although the sets $r_1[0, \bar{t}]$ and $\text{pos}^\text{tt}(\bar{t})$ are very closely related, as shown above, for any $s \in r_1[0, \bar{t}]$, $\text{val}[s]$ loses the information of $i^s$ (and hence also $A^s$) from $s$. Thus, it does not seem that Theorem 12 would imply the pigeonhole principle for $\mathcal{AR}_1$ on the topological Ramsey space $R^\text{tt}(K_1, \Sigma^*_1)$, let alone the Abstract Ellentuck Theorem for the space, which follows from our Theorem 14.

Example 2.11 in [15]. Let $H_2(n) = 2$ for $n < \omega$ and let $K_2$ consist of all FP creatures $t$ for $H_2$ such that

- $\emptyset \neq \text{dis}[t] \subseteq [m_{\text{dn}}^t, m_{\text{up}}^t]$,
- $\emptyset \neq \text{val}[t] \subseteq \text{dis}[t]$, 
- $\text{nor}[t] = \log_2(|\text{val}[t]|)$.

For $t_0, \ldots, t_n \in K_2$ with $m_{\text{dn}}^{t_i} \leq m_{\text{up}}^{t_i} \leq m_{\text{dn}}^{t_{i+1}}$, let $\Sigma_2(t_0, \ldots, t_n)$ consist of all creatures $t \in K_2$ such that

$m_{\text{dn}}^t = m_{\text{dn}}^{t_0}, \quad m_{\text{up}}^t = m_{\text{up}}^{t_n}, \quad \text{dis}[t] = \text{dis}[t_{l^*}], \quad \text{and} \quad \text{val}[t] \subseteq \text{val}[t_{l^*}], \quad \text{for some} \ l^* \leq n.

The partial ordering $\leq$ on $\text{PC}_\infty(K_2, \Sigma_2)$ is defined as follows: $\bar{t} \leq \bar{s}$ if and only if there is a sequence $(u_n)_{n<\omega}$ of finite subsets of $\omega$ such that $\max(u_n) < \min(u_{n+1})$ and for each $n < \omega$, $t_n \in \Sigma_2(\bar{s} \mid u_n)$, where $\bar{s} \mid u_n$ denotes the sequence $(s_i : i \in u_n)$. For $\bar{t} \in \text{PC}_\infty(K_2, \Sigma_2)$, $\text{pos}^\text{tt}(\bar{t})$ is defined to be $\bigcup\{\text{val}[t_n] : n < \omega\}$.

Roslanowski and Shelah proved that $(K_2, \Sigma_2)$ is a loose FFCC pair for $H_2$ which is simple except omitting and has bigness. Thus, the following Observation 2.8 (3) in [15] applies to this example to yield the following.

Proposition 17 (Roslanowski/Shelah, [15]). For any $\bar{t} \in \text{PC}_\infty(K_2, \Sigma_2)$ and any coloring $d : \text{pos}(\bar{t}) \to l$, for some $l \geq 1$, there is an $\bar{s} \leq \bar{t}$ such that $c$ is constant on $\text{pos}(\bar{s})$.

We point out that this is the same statement as Conclusion 3.10 in [15], the only difference being the hypotheses on the creating pair.

Definition 18 (The space $(\mathcal{R}(\text{PC}_\infty(K_2, \Sigma_2), \leq, r))$. Let

$\mathcal{R}(\text{PC}_\infty(K_2, \Sigma_2)) = \{\bar{s} \in \text{PC}_\infty(K_2, \Sigma_2) : \forall l < \omega, |\text{val}[t_l]| = l + 1\},$

with its inherited partial ordering. Abbreviate this space by $\mathcal{R}(K_2, \Sigma_2)$.

Theorem 19. $(\mathcal{R}(K_2, \Sigma_2), \leq, r)$ is a topological Ramsey space which is dense in the partial ordering of all pure candidates $\text{PC}_\infty(K_2, \Sigma_2)$.

Proof. It is clear that $\mathcal{R}(K_2, \Sigma_2)$ forms a dense subset of $\text{PC}_\infty(K_2, \Sigma_2)$. Towards proving that A.4 holds, let $k \geq 1$ be fixed, $\bar{t} \in \mathcal{R}(K_2, \Sigma_2)$, and $c : r_k[k-1, \bar{t}] \to 2$ be a given coloring. Each $\bar{x} \in r_k[k-1, \bar{t}]$ is of the form $\bar{x} = (t_0, \ldots, t_{k-2}, x_{k-1})$, where...
with $x_{k-1} \in \Sigma_2(\bar{t} \cap j)$ for some $j \geq k - 1$, and $\text{dis}[x_{k-1}] = \text{dis}[t_j]$ and $\text{val}[x_{k-1}] \in [\text{val}[t_j]]^k$, for some $l \in [k-1,j]$. For any $\bar{v} \in \mathcal{R}(K_2, \Sigma_2)$ and $j \geq k - 1$, let

$$X(\bar{v}, j) = \{x_{k-1} : \bar{x} \in r_k[k-1, \bar{v}] \} \cap \Sigma_2(\bar{t} \cap j).$$

Define a coloring $c'$ on the members of all such $X(\bar{v}, j)$ by

$$c'(x_{k-1}) = c(t_0, \ldots, t_{k-2}, x_{k-1}).$$

The proof of A.4 proceeds by a fusion argument as follows. Letting $\bar{t}^{k-2}$ denote $\bar{t}$, for each $j \geq k - 1$, given $\bar{t}^{-1}$, Claim 20 below yields a $\bar{t} \in [j, \bar{t}^{-1}]$ such that $c'$ is constant on $X(\bar{t}, j)$. Define $\bar{s} \leq \bar{t}$ by letting $r_{k-1}(\bar{s}) = r_{k-1}(\bar{t})$ and $s_j = t_j^1$ for all $j \geq k - 1$. Then $\bar{s}$ has the property that for each $j \geq k - 1$, $c'$ is constant on $X(\bar{s}, j)$. Take a strictly increasing sequence $(j_i)_{i \geq k-1}$, (with $j_{k-1} \geq k - 1$), such that $c'$ has the same value on all $X(s_i, j_i)$. Define $\bar{w} \in [k-1, \bar{s}]$ by $r_{k-1}(\bar{w}) = r_{k-1}(\bar{t})$, and for each $i \geq k - 1$, take $w_i$ to be any member of $\Sigma_2(s_{j_i}, \ldots, s_{j_i+1})$ such that $|\text{val}[w_i]| = i + 1$. Then $\bar{w} \in [k-1, \bar{t}]$ and $c$ is constant on $r_k[k-1, \bar{w}]$.

Claim 20. For $j \geq k - 1$, given $\bar{t}^{-1}$, there is a $\bar{t} \in [j, \bar{t}^{-1}]$ such that $c'$ is constant on $X(\bar{t}, j)$.

Proof. Each $x_{k-1} \in X(\bar{t}^{-1}, j)$ is completely determined by the triple $(n, l, \text{val}[x_{k-1}])$, where $x_{k-1} \in \Sigma_2(t_j, \ldots, t_n)$ and $l \in [j, n]$ is such that $\text{val}[x_{k-1}] \in [\text{val}[t_j]]^k$. Thus, we may regard $c'$ on $X(\bar{t}^{-1}, j)$ as a coloring of triples from

$$\{(n, l, J) : j \leq l \leq n \text{ and } J \in [\text{val}[t_j]]^k\}.$$

Letting $K_l = \text{val}[t_l]$, we see that $c'$ induces a coloring $c''$ on

$$\bigcup_{j \leq l \leq n} [K_l]^k \times \prod\{K_i : j \leq i \leq n, i \neq l\}$$

as follows: For $j \leq l \leq n$, any $p_i \in K_i$ ($i \neq l$) and $J_l \in [K_l]^k$, define

$$c''(p_{k-1}, \ldots, p_l, J_l, p_{l+1}, \ldots, p_n) = c'(n, l, J_l).$$

By Theorem 3, we obtain infinite sets $L = \{l_p : p \geq j\}$, $N = \{n_p : p \geq j\}$ such that $j \leq l_j := \min(L) \leq n_j < l_{j+1} \leq n_{j+1} < \ldots$, and subsets $H_i \subseteq K_i$ such that for each $p \geq j$, $|H_{l_p}| = p + 1$, and for each $i \notin L$, $|H_i| = 1$, and moreover, $c''$ is constant on

$$\bigcup_{n \in N} \bigcup_{l \in L \cap (n+1)} [H_l]^k \times \prod\{H_i : j \leq i \leq n, i \neq l\}.$$

Take $\bar{t} \in [j, \bar{t}^{-1}]$ such that for each $p \geq j$, $t_{l_p}^1$ is the creature in $\Sigma_2(t_l - 1, \ldots, t_{l_p} - 1)$ determined by $\text{dis}[t_p] = \text{dis}[t_{l_p}^1]$, and $\text{val}[t_p] = H_{l_p}$. Then the coloring $c'$ is constant on $X(\bar{t}, j)$.

Thus, Claim 20 holds, and by the fusion argument above, along with previous remarks about A.1 - A.3 holding, $\mathcal{R}(K_2, \Sigma_2)$ is a topological Ramsey space. $\square$
Example 2.13 in [15]. Let \( N > 0 \) and \( H_N(n) = N \) for \( n < \omega \). Let \( K_N \) consist of all FP creatures \( t \) for \( H_N \) such that

- \( \text{dis}[t] = (X_t, \varphi_t) \), where \( X_t \subseteq [m^t_{dn}, m^t_{up}] \), and \( \varphi_t : X_t \to N \);
- \( \text{nor}[t] = m^t_{up} \);
- \( \text{val}[t] = \{ f \in [m^t_{dn}, m^t_{up}] : \varphi_t \subseteq f \) and \( f \) is constant on \( [m^t_{dn}, m^t_{up}] \setminus X_t \} \).

For \( t_0, \ldots, t_n \in K_2 \) with \( m^t_{up} = m^{t_0}_{dn} \), let \( \Sigma_N(t_0, \ldots, t_n) \) consist of all creatures \( t \in K_N \) such that

- \( m^t_{dn} = m^{t_0}_{dn}, m^t_{up} = m^{t_0}_{up}, X_{t_0} \cup \cdots \cup X_{t_n} \subseteq X_t \),
- for each \( l \leq n \), either \( X_t \cap [m^t_{ dn}, m^t_{ up}] = X_{ t_l } \) and \( \varphi_t \restriction [m^t_{ dn}, m^t_{ up}] = \varphi_{ t_l } \), or \( [ m^t_{dn}, m^t_{up}] \subseteq X_t \) and \( \varphi_t \restriction [m^t_{dn}, m^t_{up}] \in \text{val}[t] \).

The partial ordering \( \preceq \) on \( \text{PC}^c_\infty(K_N, \Sigma_N) \) is defined by \( t \preceq s \) if and only if there is a strictly increasing sequence \( (s_{j_n})_{n < \omega} \) such that each \( t_n \in \Sigma_N(s_{j_n}, \ldots, s_{j_{n+1}-1}) \).

Roslanowski and Shelah proved that \( (K_N, \Sigma_N) \) is a tight FFPC pair for \( H_N \) which has the \( t \)-multiadditivity and weak bigness, and is gluing [14], 2.1.7 for each pure candidate in \( \text{PC}^c_\infty(K_N, \Sigma_N) \). Thus, Conclusion 4.8 of [15] holds for this example: that is, Theorem 12 with each instance of \( \text{PC}^c_\infty(K_1, \Sigma_1) \) replaced with \( \text{PC}^c_\infty(K_N, \Sigma_N) \), holds.

We show that this forcing itself forms a topological Ramsey space. The pigeonhole principle A.4 will follow from the Hales-Jewett Theorem in [10]. This space is extremely similar to the space of infinite sequences of variable words, which Carlson showed to be a topological Ramsey space in [1], and which corresponds to the “loose” version. We point out that Conclusion 4.8 of [15] for this example does not follow from \( \text{PC}^c_\infty(K_N, \Sigma_N) \) being a topological Ramsey space, since members \( s \in r_1[0, \bar{t}] \) may have \( \text{val}[s] \) of any cardinality.

Theorem 21. \( (\text{PC}^c_\infty(K_N, \Sigma_N), \preceq, r) \) is a topological Ramsey space.

Proof. Let \( k \geq 2 \) and \( \bar{t} \in \text{PC}^c_\infty(K_N, \Sigma_N) \) be given. There is a one-to-one correspondence \( \sigma \) between \( r_k[k-1, \bar{t}] \) and the set of finite variable words on alphabet \( N \): For \( (l_0, \ldots, t_{k-2}, s) \in r_k[k-1, \bar{t}] \), let \( \sigma(s) \) denote the variable word \( (l_{k-1}, \ldots, l_0) \) where \( m \geq k-1 \) is such that \( s \in \Sigma_N(t_{k-1}, \ldots, t_m) \) and for each \( i \in [k-1, m] \), \( l_i \in N \) if and only if \( \varphi_s \restriction [m^i_{dn}, m^i_{up}] \subseteq X_{t_i} \) and \( l_i = v \) if and only if \( X_s \cap [m^i_{dn}, m^i_{up}] = X_{t_i} \).

Given a coloring \( c : r_k[k-1, \bar{t}] \to 2 \), let \( c' \) color the collection of all variable words on alphabet \( N \) by \( c'(\sigma(s)) = c(t_0, \ldots, t_{k-2}, s) \). By the Hales-Jewett Theorem, there is an infinite sequence of variable words \( (x_i)_{i < \omega} \) such that \( c' \) is constant on all variable words of the form \( x_{i+1}[\lambda_0] \cdots x_i[\lambda_0] \), where each \( \lambda_j \in N \cup \{ v \} \) and at least one \( \lambda_j = v \). For each \( i \geq k-1 \), let \( l(i) = |x_i| \), the length of the word \( x_i \). Let \( m_0 = k-1 + l(k-1) \), and given \( i < \omega \) and \( m_i \), let \( m_{i+1} = m_i + l(i) \). Let \( l_0, \ldots, l_{(l(i)-1)} \) denote \( x_i \). Define \( s_{k-1} \) to be the member of \( \Sigma_N(t_{k-1}, \ldots, t_{m_{k-1}-1}) \) such that \( \sigma(s_{k-1}) = x_0 \), and in general, for \( i \geq 1 \) define \( s_{k-1+i} \) to be the member of \( \Sigma_N(t_{m_{k-1}}, \ldots, t_{m_i-1}) \) such that \( \sigma(s_{k-1+i}) = x_i \). Letting \( \bar{s} = r_k[0, \bar{t}] \setminus (s_{k-1}, s_{k-2}, \ldots) \), it is routine to check that \( c \) is monochromatic on \( r_k[k-1, \bar{s}] \). Hence, A.4 holds for \( k \geq 2 \).

A fusion argument identical to the proof of Claim 16 yields the following.

Claim 22. Given \( \bar{t} \in \text{PC}^c_\infty(K_N, \Sigma_N) \) and colorings \( c_k : A(R_k+1)[\bar{t}] \to l_k \), for some \( l_k \geq 1 \), there is an \( \bar{s} \leq \bar{t} \) in \( \text{PC}^c_\infty(K_N, \Sigma_N) \) such that \( m^s_{dn} = m^t_{dn} \), \( c_0 \) is constant on \( r_1[0, \bar{s}] \cap \Sigma_N(\bar{s}) \), and for each \( k \geq 1 \), the coloring \( c_k \) on \( r_{k+1}[k, \bar{s}] \) is constant.
Finally, to prove A.4 for $\mathcal{AR}_1$, let $c : r_1[0, \bar{t}] \to 2$ be given. For each $k < \omega$, define a coloring $c_k : \mathcal{AR}_{k+1}[\bar{t}] \to 2$ by $c_k(x_0, \ldots, x_k) = c(x_k)$. Take $\bar{s} \leq \bar{t}$ satisfying 

Claim 22. There is a strictly increasing sequence $(k_j)_{j<\omega}$ such that the color of $c_{k_j}$ on $r_{k_j+1}[k_j, \bar{s}]$ is the same for all $j < \omega$. Take $\bar{v} \leq \bar{s}$ satisfying that for each $j < \omega$, $m_{\bar{v}_j}^{dn} = m_{\bar{s}_j}^{dn}$ and $m_{\bar{v}_j}^{up} = m_{\bar{s}_j}^{up}+1$, and $v_j \in \Sigma^N(s_{k_j}, \ldots, s_{k_{j+1}})$. Then $c$ is constant on $r_1[0, \bar{v}]$.

Thus, A.4 holds, and hence the Theorem holds.

5. Remarks and Further Lines of Inquiry

Whenever a forcing contains a topological Ramsey space as a dense subset, this has implications for the properties of the generic extension and provides as well Ramsey-theoretic techniques for streamlining proofs. Although this note only showed that the pure candidates for three examples of creature forcings contain dense subsets forming topological Ramsey spaces, the work here points to and lays some groundwork for several natural lines of inquiry.

One obvious line of exploration is to develop stronger versions and other variants of Theorem 3 to obtain the pigeonhole principle for the pure candidates for other creating pairs, in particular for Example 2.12 in [15]. Another is to develop this theory for the loose candidates, as we only considered tight types here. A deeper line of inquiry is to determine the implications that the existence of a topological Ramsey space dense in a collection of pure candidates for a creating pair has for the forcing notion (with stems) generated by that creating pair.

The topological Ramsey spaces considered here force ultrafilters on base set $K$, a set of creatures, which in turn generate ultrafilters on a countable set of finite functions $\mathcal{F}_H$. The work here yields partition theorems of Rosłanowski and Shelah in [15] for two of examples considered in Section 4. It will be interesting to see how their Glazer methods interact with the product tree Ramsey methods more abstractly. The hope is that this article has piqued the reader’s interest to investigate further the connections between creature forcings and topological Ramsey spaces as such investigations will likely will lead to new Ramsey-type theorems and new topological Ramsey spaces, while adding to the collection of available techniques and streamlining approaches to at least some genres of the myriad of creature forcings.

References

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University of Denver, Department of Mathematics, 2280 S Vine St, Denver, CO 80208, USA

E-mail address: natasha.dobrinen@du.edu

URL: [http://web.cs.du.edu/~ndobrine](http://web.cs.du.edu/~ndobrine)