Banach spaces from barriers in high dimensional Ellentuck spaces

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Spring Topology and Dynamics Conference, 2017

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A Banach space is a complete normed vector space.

c_0 is the space of sequences $(x_n)_{n=1}^{\infty}$ tending toward 0 with the sup norm

$$\|x\|_\infty = \sup_n |x_n|.$$

ℓ_p is the space of sequences $(x_i)_{i=1}^{\infty}$ such that

$$\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty.$$

ℓ_∞ is the space of all bounded sequences with the sup norm.

Theorem. (Tsirelson) There is a reflexive Banach space with an unconditional basis not containing c_0 or ℓ_p for $1 \le p < \infty$.

Tsirelson's construction method can be generalized to any barrier.

Argyros and Deliyanni made a systematic study of this construction method for all barriers.

We concentrate on the finite rank barriers $[\omega]^d$ and their compact closures $\mathcal{A}_d = [\omega]^{\le d}$.

For a finite set $E \subseteq \omega$ and a vector $x = \sum_n x_n e_n$,

$$Ex = E\left(\sum_n x_n e_n\right) := \sum_{n \in E} x_n e_n.$$

Low complexity Tsirelson-type spaces $T(\mathcal{A}_d, \theta)$

c_{00} is the set of all infinite sequences $(x_n)_{n=1}^{\infty}$ where all but finitely many x_i are zero. Let $1 \leq d < \omega$ and $0 < \theta < 1$. $\mathcal{A}_d = [\omega]^{\leq d}$.

The space $T(\mathcal{A}_d, \theta)$ is the completion of c_{00} with the norm $\| \cdot \|_{T(\mathcal{A}_d,\theta)}$ defined recursively as follows. Given $x = \sum_{n=1}^{\infty} x_n e_n \in c_{00}$,

$$|x|_0 = \max_n |x_n|.$$

$$|x|_{j+1} = \max\left\{ |x|_j, \theta \max\{\sum_{i=1}^{m} |E_i x|_j : 1 \leq m \leq d \text{ and } E_1 < E_2 < \cdots < E_m\} \right\}$$

$$\|x\|_{T(\mathcal{A}_d,\theta)} = \sup_{j<\omega} |x|_j.$$

Theorem. (Bellenot) If $d\theta > 1$ and p is such that $d^{\frac{1}{p}} = d\theta$, then for each $x \in T(\mathcal{A}_d, \theta)$,

$$\frac{1}{2d}\|x\|_p \leq \|x\|_{T(\mathcal{A}_d,\theta)} \leq \|x\|_p.$$

Thus, $T(\mathcal{A}_d, \theta)$ is isomorphic to ℓ^p.

$\mathcal{A}_d = [\omega]^{\leq d}$ is the closure under inclusion of the d-dimensional barrier on the Ellentuck space.

Motivated by a problem regarding the Tukey structures of ultrafilters, we constructed new topological Ramsey spaces in [D1] and [D2] which turn out to be high dimensional analogues of the Ellentuck space.

Ultrafilters, Tukey, and high dimensional Ellentuck spaces

$\mathcal{P}(\omega)/\mathrm{Fin}$ forces a Ramsey ultrafilter $\mathcal{U}$.

An equivalent forcing is Mathias forcing mod finite: $([\omega]^\omega, \subseteq^*)$.

This is exactly forcing with the Ellentuck space mod finite.

The ideal $\mathrm{Fin} \otimes \mathrm{Fin}$, also denoted $\mathrm{Fin}^{\otimes 2}$, is the set of all $X \subseteq \omega \times \omega$ such that for all but finitely many n, the n-th fiber of X is finite.
$\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$ forces a non-p-point $\mathcal{U}_2$ satisfying the partition relation

$$\mathcal{U}_2 \to (\mathcal{U}_2)^2_{I,4}.$$

The projection $\pi_1(\mathcal{U}_2)$ to the first coordinates is Ramsey and is generic for $\mathcal{P}(\omega)/\mathrm{Fin}$.

The two dimensional Ellentuck space $\mathcal{E}_2$

In [D1] we constructed a new topological Ramsey space $\mathcal{E}_2$ which is dense in the forcing $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$.

The purpose was to answer a question left open in [Blass/D./Raghavan] as to the exact Tukey structure below $\mathcal{U}_2$. In [D1] we found that the Tukey structure below $\mathcal{U}_2$ is simply a chain of length 2.

$\mathcal{E}_2$ turned out to be the two dimensional extension of the Ellentuck space. In fact, we constructed α-dimensional Ellentuck spaces for each countable ordinal α in [D1] and [D2] which are dense in forcings which produce ever more complex ultrafilters (see [D3]).

As topological Ramsey spaces, they come equipped with notions of barriers (see [Todorcevic]).

Motivating Question

The project in this talk was initiated by the following question.

Question. (D.) What kinds of Banach spaces can be constructed by extending Tsirelson's construction to general topological Ramsey spaces?

A natural starting point for answering this question is the extensions of the Ellentuck space to higher dimensions.

Results

$\mathcal{E}_k$ denotes the k-dimensional Ellentuck space (defined soon).

For each pair $1 \leq d < \omega$ and $0 < \theta < 1$, $T_k(d, \theta)$ denotes the Banach space constructed from $\mathcal{E}_k$ using sequences $E_1 < \cdots < E_d$ of admissible sets (defined soon).

Theorem. (ADGM) Given k, d, θ, $T_k(d, \theta)$ is a Banach space with the following properties:

1. $T_k(d, \theta)$ is ℓ_p saturated.
2. $T_k(d, \theta)$ contains copies of ℓ_∞^n, the bound being the same for all n.
3. For $j < k$, $T_j(d, \theta)$ is not isomorphic to $T_k(d, \theta)$.
4. For $j < k$, $T_k(d, \theta)$ contains subspaces isometric to $T_j(d, \theta)$.

The 2-dimensional Ellentuck space $\mathcal{E}_2$

$(\mathcal{E}_2, \subseteq^{\mathrm{Fin}^{\otimes 2}})$ is forcing equivalent to $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$.

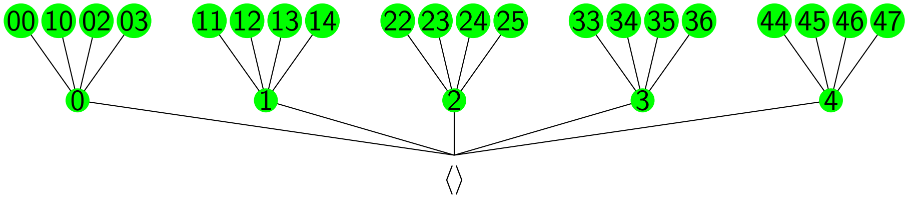
$\omega^{\not\downarrow \leq 2}$ denotes the set of all non-decreasing sequences of natural numbers of length less than or equal to two.

Let $\prec$ well-order $\omega^{\not\downarrow \leq 2}$ in order type ω as follows:

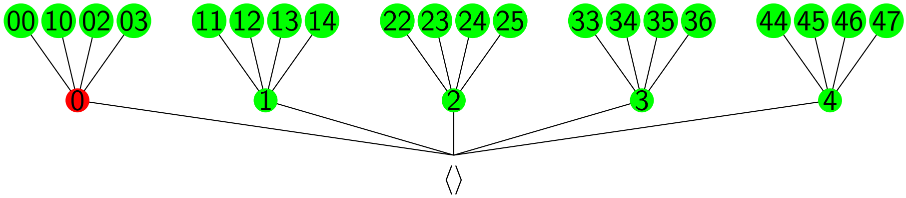
$$(s_0, s_1) \prec (t_0, t_1) \Leftrightarrow (s_1 < t_1) \text{ or } (s_1 = t_1 \text{ and } s_0 < t_0).$$

Members of $\mathcal{E}_2$ are subsets $X \subseteq \omega^{\not\downarrow 2}$ which as trees have the same structure as $\omega^{\not\downarrow 2}$ with respect to both tree structure and $\prec$ ordering.

$\omega^{\not\downarrow \leq 2}$ has lexicographic order-type ω^2



The well-order $(\omega^{\not\leq 2}, \prec)$



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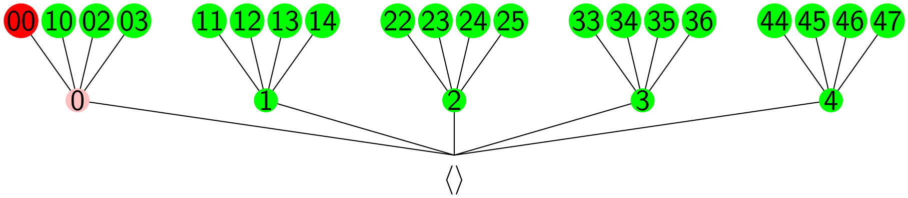

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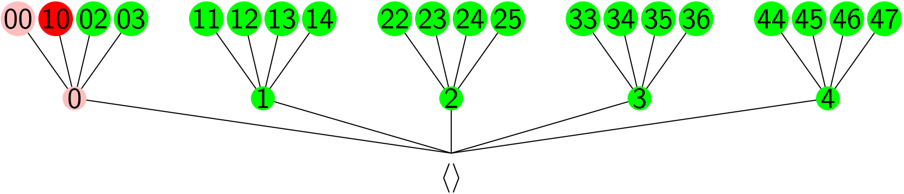

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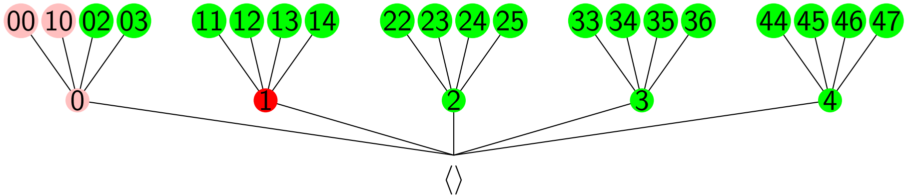

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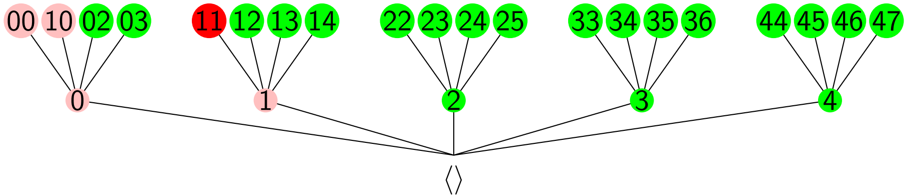

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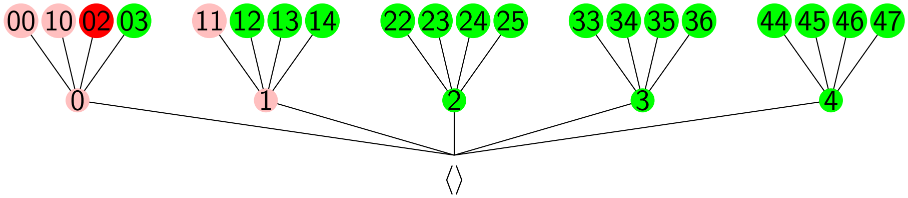

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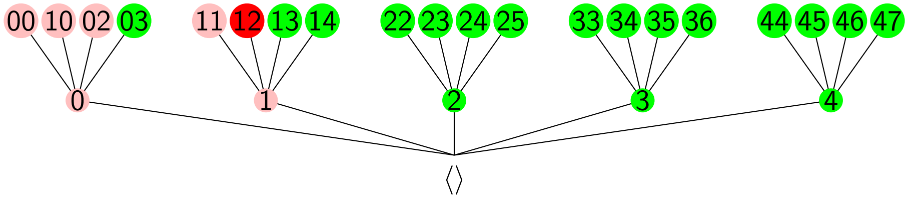

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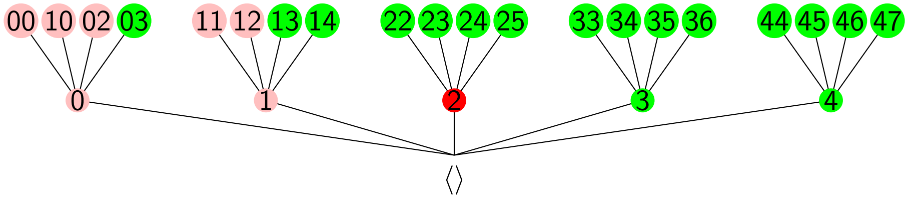

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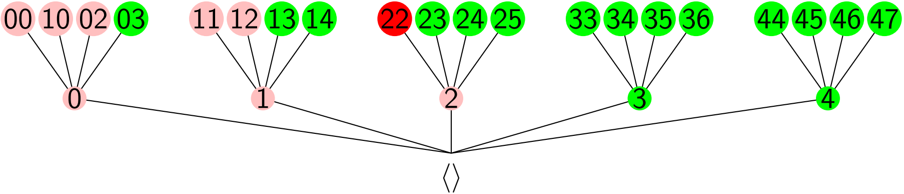

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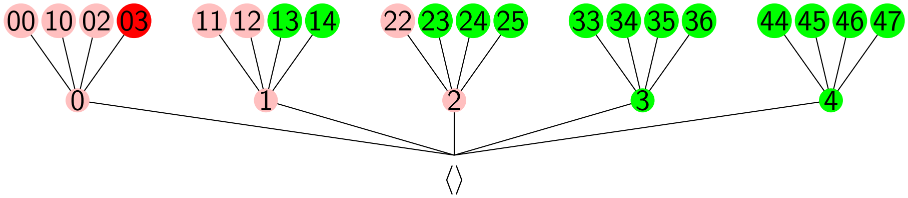

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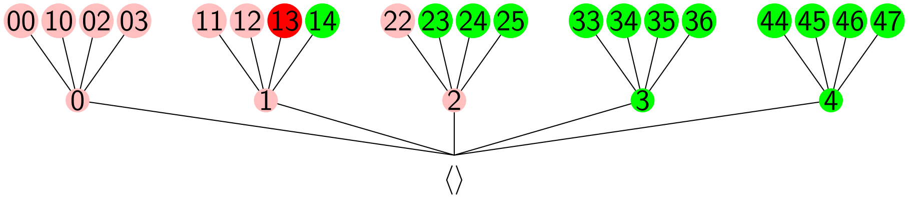

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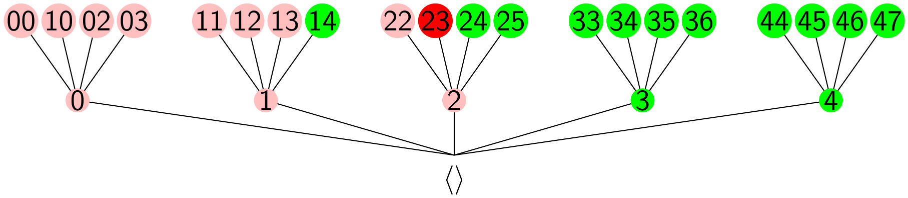

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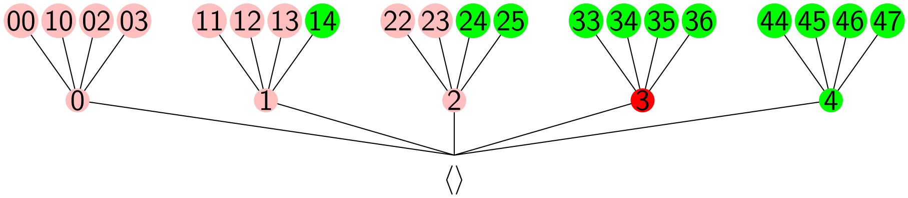

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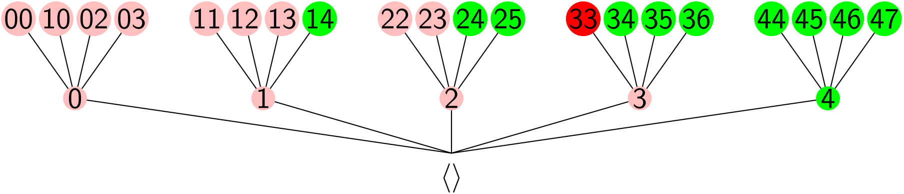

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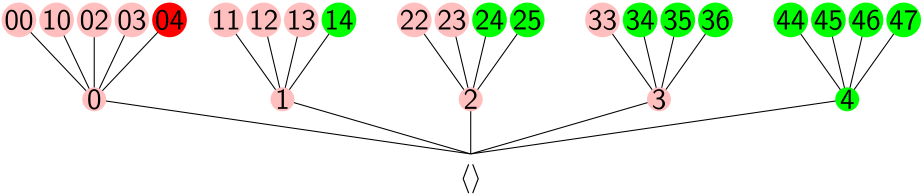
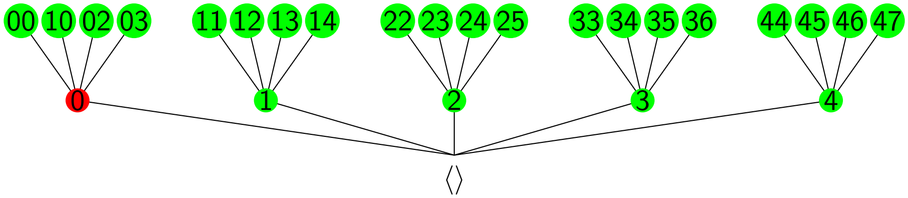


Example: $X \in \mathcal{E}_2$



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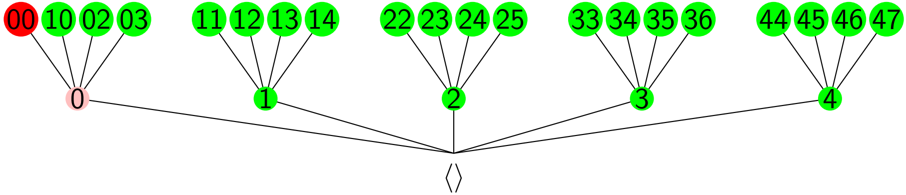

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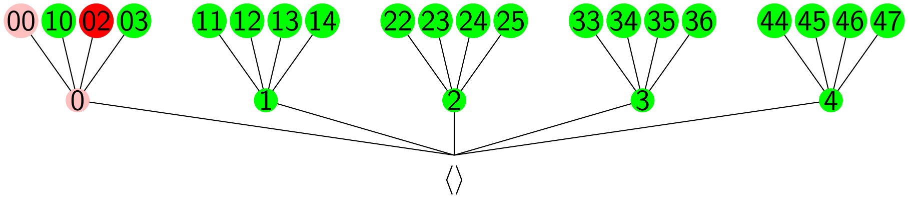

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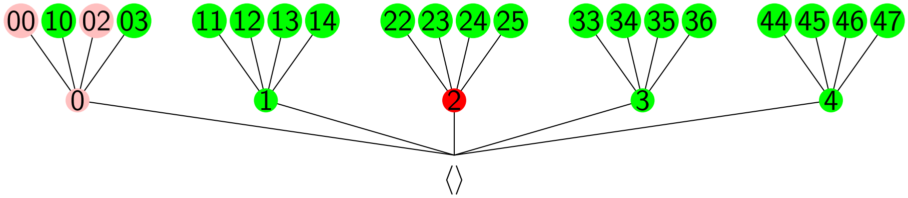

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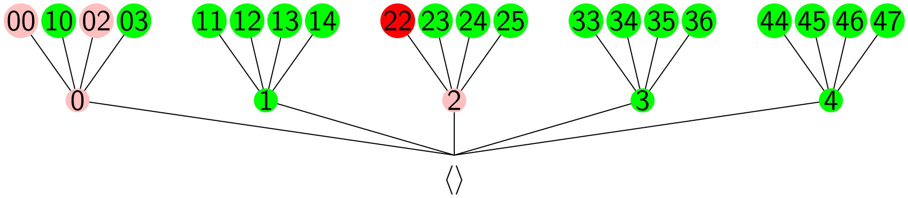

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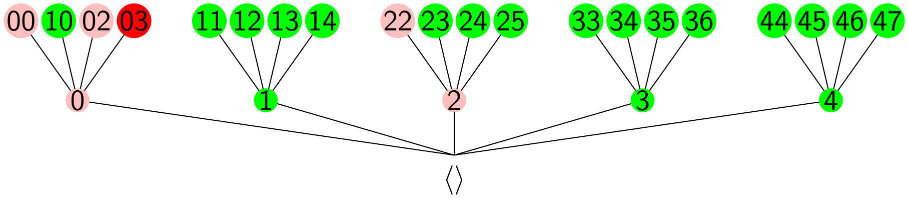

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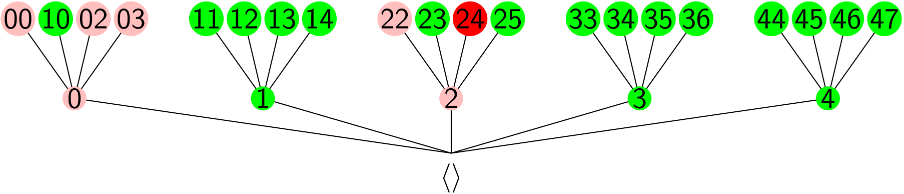

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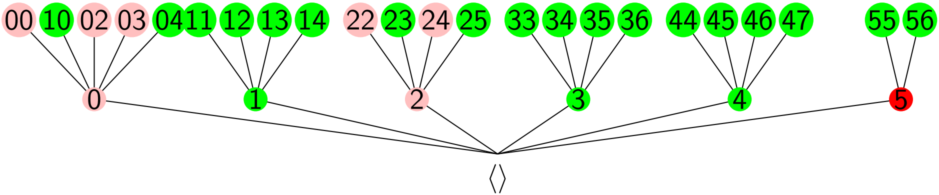

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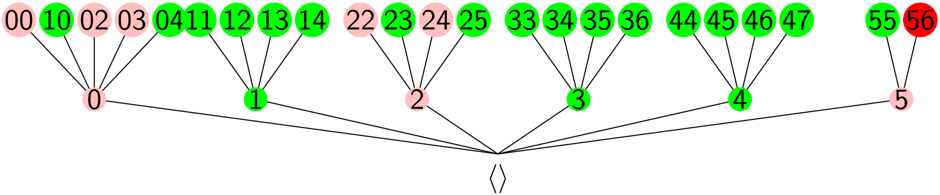
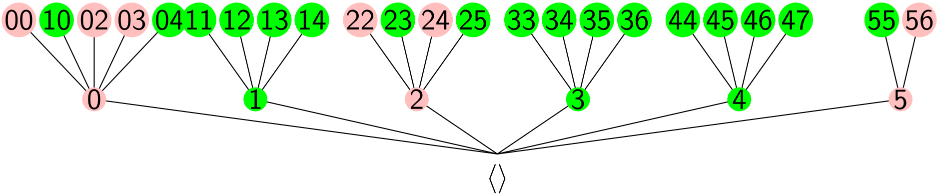
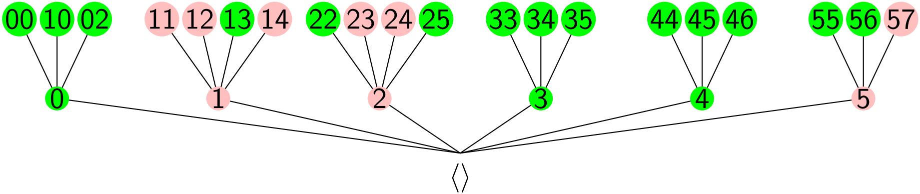


The pink dots are a 6-th approximation to $X \in \mathcal{E}_2$.



A 6-th approximation to a different member of $\mathcal{E}_2$.



The collection of all these types of sets is the notion for $\mathcal{E}_2$ corresponding to the barrier $[\omega]^6$ in the Ellentuck space.

Finite dimensional barriers on $\mathcal{E}_2$

The i-th dimensional barrier $\mathcal{AR}^2_i$ on $\mathcal{E}_2$ is the collection of all i-th approximations of members of $\mathcal{E}_2$.

This is the analogue for $\mathcal{E}_2$ of the i-dimensional barrier $[\omega]^i$ on the Ellentuck space.

$$\mathcal{AR}^2 = \bigcup_{i<\omega} \mathcal{AR}^2_i.$$

There are several choices to make for extending Tsirelson's construction to high dimensional Ellentuck spaces:
The admissible sets could be in the closure of a barrier or simply finite sets.
The endpoints separating the sequence of admissible sets could be either in $\bigcup_{m\leq d} \mathcal{AR}^2_m$ or simply finite sets of size $\leq d$.

The Banach space $T_2(d, \theta)$

We define $T_2(d, \theta)$ as the completion of c_{00} under the norm defined using sequences

$$v_1 \preceq E_1 \prec \cdots \prec v_m \preceq E_m$$

where $m \leq d$, the collection $\{v_i : 1 \leq i \leq m\}$ is simply a set, and each $E_i \in \mathcal{AR}^2$.

The Banach space $T_2(d, \theta)$

The basis for the space consists of the non-decreasing sequences of length two, ordered by $\prec$. Denote these as e_n, $n < \omega$.

Let $x = \sum_{n=1}^{\tilde{n}} x_n e_n$ be a member of c_{00}. Define $|x|_0 = \max_n |x_n|$.

$$|x|_{j+1} = \max \left\{ |x|_j, \theta \max\{\sum_{i=1}^{m} |E_i x|_j : (E_i)_{i=1}^{m} \text{ is admissible, } m \le d\} \right\}.$$

$$\text{Define } \|x\|_{T(\mathcal{A}_d, \theta)} = \sup_{j<\omega} |x|_j.$$

Theorem. (ADGM) Given k, d, θ, $T_k(d,\theta)$ is a Banach space with the following properties:

1. $T_k(d,\theta)$ is ℓ_p saturated.
2. $T_k(d,\theta)$ contains copies of ℓ_∞^n, the bound being the same for all n.
3. For $j < k$, $T_j(d,\theta)$ is not isomorphic to $T_k(d,\theta)$.
4. For $j < k$, $T_k(d,\theta)$ contains subspaces isometric to $T_j(d,\theta)$.

$T_1(d, \theta)$ embeds isometrically as a subspace of $T_2(d, \theta)$

Let $T_2(\theta, d)[(0)]$ be the subspace of $T_2(\theta, d)$ generated by the basis elements $\{(0, n) : n < \omega\}$.

Given $y = \sum_{n=1}^{r} x_n(0, j_n)$, let $\varphi(y) = \text{tr}_0(y) = \sum_{n=1}^{r} x_n(j_n)$. Then

$$\|y\|_{T_2(\theta,d)[(0)]} = \|\varphi(y)\|_{T_1(\theta,d)}.$$

This follows from the fact that the trace above (0) of any admissible sequence in $\mathcal{E}_2$ is an admissible sequence in $\mathcal{E}_1$.

Thus, φ maps $T_2(\theta, d)[(0)]$ isometrically to $T_1(\theta, d)$.

This is the idea behind the more general result that $T_{k+1}(d, \theta)$ contains a subspace isometric to $T_k(d, \theta)$.

Ex. The norms on $T_1(2, 3/4)$ and $T_2(2, 3/4)$ are different

Let $x = 2(0,0) + 1(1,1) + 1(0,2)$ in $T_2(2, 3/4)$.
Then

$$\|x\|_{T_2(2,3/4)} = |x|_2 = \frac{3}{4}\left(\frac{3}{4}(2+1)+1\right) = \frac{39}{16}.$$

Let $z = 2e_i + e_k + e_l$ in $T_1(2, 3/4)$ for any $i < k < l$.
Then

$$\|x\|_{T_1(2,3/4)} = |x|_2 = \frac{3}{4}\left(2+\frac{3}{4}(1+1)\right) = \frac{21}{8}.$$

Thus, no vector in $T_1(2, 3/4)$ formed with three basis elements and the coefficients $2, 1, 1$ has norm equal to $\|x\|_{T_2(2,3/4)}$.

In general, the norm for any vector in $T_{k+1}(d, \theta)$ is no larger than the norm for a similar vector in $T_k(d, \theta)$.

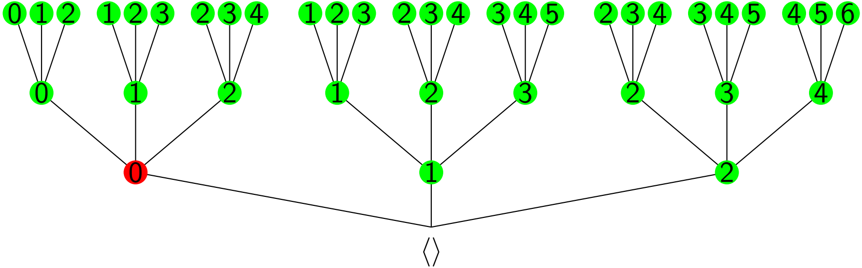
The 3-dimensional Ellentuck space $\mathcal{E}_3$

The the 3-dimensional Ellentuck space is forcing equivalent to $\mathcal{P}(\omega^3)/\mathrm{Fin}^{\otimes 3}$.

This produces and ultrafilter $\mathcal{U}_3$ which projects to the ultrafilters $\mathcal{U}_2$ and the Ramsey ultrafilter $\mathcal{U}_3$.

The space $\mathcal{E}_3$ was built in order to obtain a precise analysis of the Rudin-Keisler and Tukey structures below $\mathcal{U}_3$.

The structure of members in $\mathcal{E}_3$



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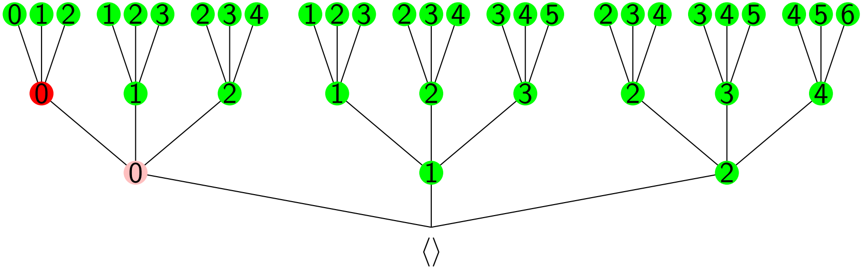

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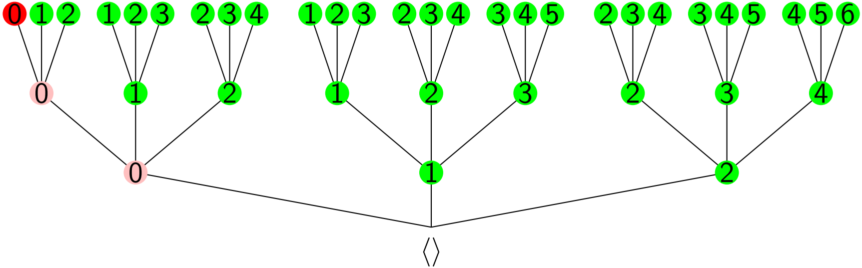

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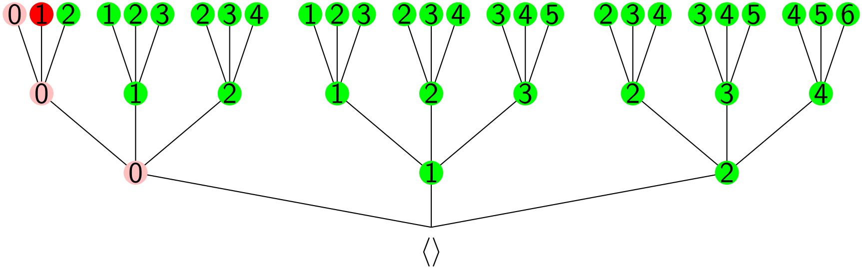

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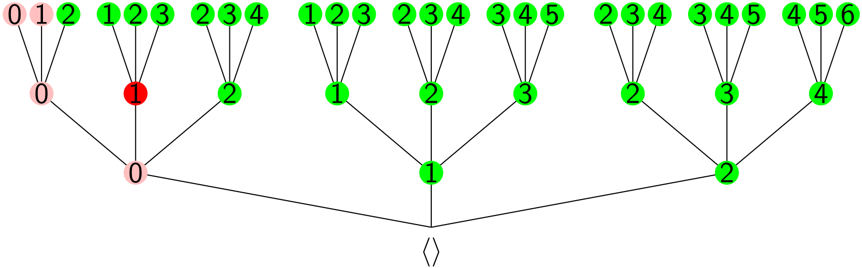

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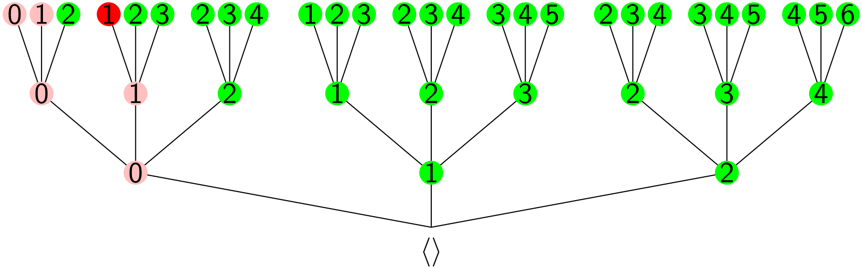

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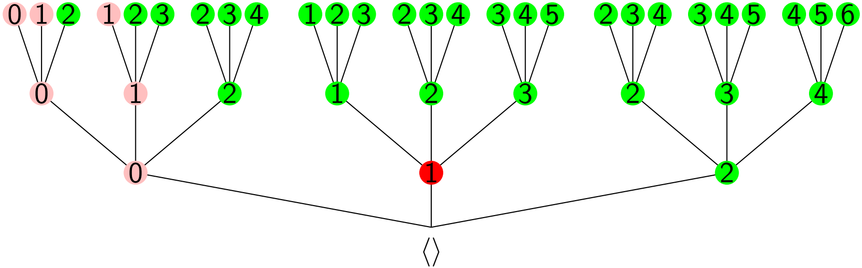

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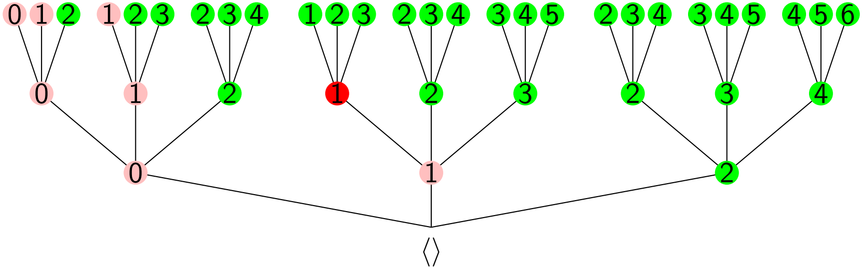

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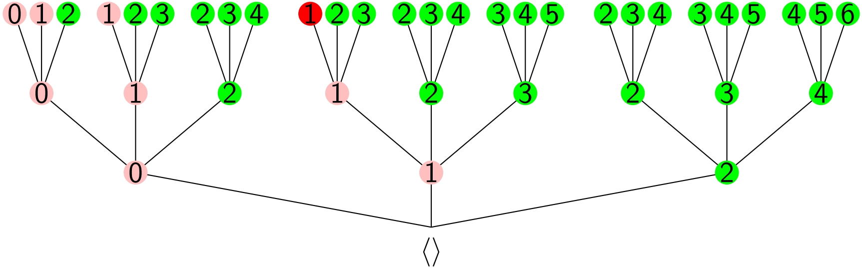

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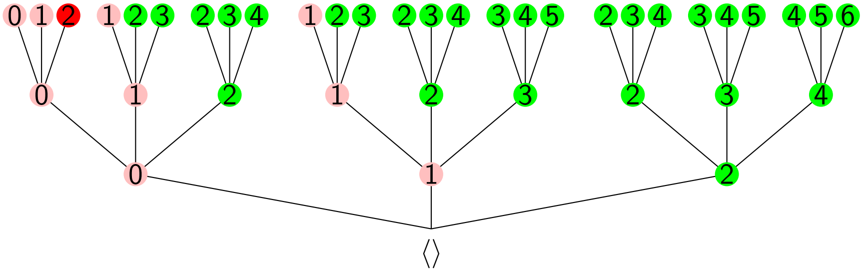

The $\mathcal{E}_k$ for $k \geq 3$ continues this kind of structure.
These topological Ramsey spaces are forcing equivalent to $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$, adding a hierarchy of non-p-point ultrafilters satisfying weaker partition relations.

Future Directions

Given that the norms the $T_k(d,\theta)$ are non-increasing and, in some places, decreasing as k increases, it seems plausible that more general spaces, using higher dimensional Ellentuck spaces and/or higher dimensional barriers, could provide interesting examples regarding distortion.

What other properties or new types of Banach spaces and be obtained by applying the barrier construction of a norm?

How do the Banach spaces differ depending on which of the three possible combinations for endpoints and admissible sets are chosen for defining the norm?

References

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