Tutorial: Ramsey theory in Forcing - Day 3

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Day 3: Forcing in Ramsey Theory

Forcing in Ramsey Theory on Trees
and Applications to Relational Structures

Day 3 Outline

The focus today is on

1. The Halpern-Läuchli Theorem and a forcing proof.
2. Milliken's topological Ramsey space of strong trees.
3. Applications to finite Ramsey degrees for the Rado graph.
4. A new almost topological Ramsey space of strong triangle-free trees.
5. Finite Ramsey degrees for the universal triangle-free graph.
6. Extensions to structures of measurable cardinality.

Strong Trees

A tree $T \subseteq 2^{<\omega}$ is a **strong subtree** if there is an infinite set $L \subseteq \omega$ such that for each node $t \in T$, t splits iff $\mathrm{lh}(t) \in L$.

The Halpern-Läuchli Theorem (Strong Tree Version)

For a tree $T \subseteq 2^{<\omega}$ and $l < \omega$, let $T(l)$ denote $T \cap 2^l$.

Thm. (Halpern-Läuchli) Given any strong trees $T_i \subseteq 2^{<\omega}$, $i < d$, $L \in [\omega]^\omega$ the levels of splitting nodes in each T_i, and a coloring

$$c : \bigcup_{l \in L} \prod_{i<d} T_i(l) \to 2,$$

there are strong subtrees $S_i \subseteq T_i$ and an $L' \in [L]^\omega$ which is the set of the splitting levels in each S_i such that c is monochromatic on

$$\bigcup_{l' \in L'} \prod_{i<d} S_i(l').$$

Harrington's forcing proof of Halpern-Läuchli

(Proof as outlined for me by Laver)

Let $d \geq 1$ and for each $i < d$, let T_i be a strong subtree of $2^{<\omega}$. Fix a coloring

$$c : \bigcup_{l \in L} \prod_{i<d} T_i(l) \to 2.$$

Thm. (Erdős-Rado) For $r \geq 0$ finite and μ an infinite cardinal,

$$\beth_r(\mu)^+ \to (\mu^+)^{r+1}_{\mu}.$$

Let $\kappa = \beth_{2d-1}(\aleph_0)^+$. Then

$$\kappa \to (\aleph_1)^{2d}_{\aleph_0}.$$

The following forcing notion $\mathbb{P}$ will add κ many generic paths $\dot{b}_{i,\alpha}$, $\alpha < \kappa$, through each T_i, $i < d$.

$$p \in \mathbb{P} \text{ iff is a partial function } p \text{ with } \mathrm{dom}(p) = d \times \vec{\delta}_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i(l_p)$, for each $i < d$, where $l_p \in \omega$.

$q \leq p$ if and only if either

1. $l_q = l_p$ and $q \supseteq p$; or else
2. $l_q > l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and $q(i,\delta) \supset p(i,\delta)$, for each pair $(i,\delta) \in d \times \vec{\delta}_p$.

Remark. This is essentially Cohen forcing but on the trees.

Let $\dot{\mathcal{U}}$ be a name for an ultrafilter on ω and
let $\dot{b}_{\vec{\alpha}} = (\dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_d-1})$.

For each $\vec{\alpha} = (\alpha_0, \dots, \alpha_{d-1}) \in [\kappa]^d$, choose a condition $p_{\vec{\alpha}} \in \mathbb{P}$ such that

1. $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$,
2. $p_{\vec{\alpha}} \Vdash$ "$\exists \varepsilon \in 2$ such that $c(\dot{b}_{\vec{\alpha}}(l)) = \varepsilon$ for $\dot{\mathcal{U}}$ many l",
3. $p_{\vec{\alpha}}$ decides a value for ε, call it $\varepsilon_{\vec{\alpha}}$,
4. $c(p_{\vec{\alpha}}(i, \alpha_i) : i < d) = \varepsilon_{\vec{\alpha}}$.

Claim. There are $K_0 < \cdots < K_d$ infinite subsets of κ such that the set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

There is one $\varepsilon_* < 2$ such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,
$p_{\vec{\alpha}}$ forces $c(\dot{b}_{\vec{\alpha}}(l)) = \varepsilon_*$ on $\dot{\mathcal{U}}$ many levels l.

There are nodes t_i^* such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $p_{\vec{\alpha}}(i, \alpha_i) = t_i^*$.

This follows from a judicious coloring with ω many colors and an application of the Erdős-Rado Theorem.

Make a coloring f on $[\kappa]^{2d}$ which codes all the information we need:

Let $\mathcal{I}$ denote the collection of all functions $\iota : 2d \to 2d$ such that $\iota \restriction \{0, 2, \ldots, 2d-2\}$ and $\iota \restriction \{1, 3, \ldots, 2d-1\}$ are strictly increasing sequences and $\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}$.

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, letting $\vec{\alpha} = \iota_e(\vec{\theta}\,)$, $\vec{\beta} = \iota_o(\vec{\theta}\,)$, and $k_{\vec{\alpha}} := |\vec{\delta}_{\vec{\alpha}}|$, let

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}},, \langle \langle i, j \rangle : i < d,\ j < k_{\vec{\alpha}},\ \text{and } \vec{\delta}_{\vec{\alpha}}(j) = \alpha_i \rangle, \\ \langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle, \langle \langle p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) : j < k_{\vec{\beta}} \rangle : i < d \rangle, \\ \langle \langle j, k \rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle. \end{aligned} \tag{1}$$

Let $f(\vec{\theta})$ be the sequence $\langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$, where $\mathcal{I}$ is given some fixed ordering. By the Erdős-Rado Theorem, there is $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

From the homogeneity of f on $K \in [\kappa]^{\aleph_1}$, one can prove the claim.

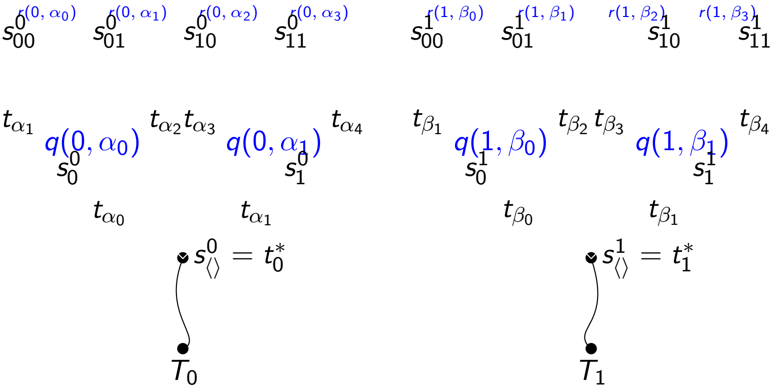
Claim. There are $K_0 < \cdots < K_d$ infinite such that the set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

There is one $\varepsilon_* < 2$ such that for all $\vec{\alpha} \in \prod_{i<d} K_i$, $p_{\vec{\alpha}}$ forces $c(\dot{b}_{\vec{\alpha}}(l)) = \varepsilon_*$ on $\dot{\mathcal{U}}$ many levels l.

There are nodes t_i^* such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $p_{\vec{\alpha}}(i, \alpha_i) = t_i^*$.

Now use conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ to help build strong subtrees S_i extending the node t_i^*, $i < d$, which have the same splitting levels and all have the same c-color ε_* on each product of level sets.

Constructing S_i in T_i to satisfy H-L Thm.



Take $q \leq p_{(\alpha_0,\beta_0)}$, $p_{(\alpha_0,\beta_1)}$, $p_{(\alpha_1,\beta_0)}$, $p_{(\alpha_1,\beta_1)}$ with each $q(0,\alpha_i) \sqsupseteq t_{\alpha_i}$, and $q(1,\beta_j) \sqsupseteq t_{\beta_j}$, and deciding color ε_* for all pairs of nodes of the form $\{q(0,\alpha_i), q(1,\beta_j) : i,j < 2\}$. Let $s^0_i = q(0,\alpha_i)$ and $s^1_j = q(1,\beta_j)$.

Milliken's topological Ramsey space of strong trees ($d = 1$)

The **Milliken space** is the triple $(\mathcal{M}, \leq, r)$, where

1. $\mathcal{M}$ consists of all strong subtrees $T \subseteq 2^{<\omega}$,
2. $S \leq T$ iff S is a subtree of T,
3. The n-th restriction of a tree T is the initial subtree $r_n(T) = \{t \in T : t \text{ there are } < n \text{ splitting nodes below } t\}$.

$\mathcal{AR}_n$ is the collection $\{r_n(T) : T \in \mathcal{M}\}$.

$\mathcal{AR}$ is the collection of all finite strong subtrees.

On board - show pictures of $r_{k+1}[k, T]$ for various $k < \omega$.

Notice that the proof of **Axiom A.4** is really an application of the Halpern-Läuchli Theorem.

In the example of **A.4** for $r_2[1, T]$, we are applying the Halpern-Läuchli Theorem for $d = 2$ trees, one tree above each of the immediate successors of the maximal nodes in $r_1(T)$.

Exercize. Use the Halpern-Läuchli Theorem to prove **Axiom A.4** for the Milliken space.

Milliken's Theorem

For $T \in \mathcal{M}$, let $\mathcal{AR}_k|T$ denote $\{r_k(S) : S \leq T\}$.

The following is a special case of the Abstract Nash-Williams Theorem for the Milliken space:

Thm. (Milliken) For each $k < \omega$, $T \in \mathcal{M}$ and coloring $c : \mathcal{AR}_k|T \to 2$, there is an $S \leq T$ such that c is one color on $\mathcal{AR}_k|S$.

Milliken's Theorem was one of the key ingredients of Sauer's proof that the Rado graph has finite Ramsey degrees.

The Rado graph

The Rado graph is the random graph on ω many vertices.

The Rado graph is the universal countable graph.

The Rado graph is the homogeneous countable graph.

The Rado graph is the Fraïssé limit of the Fraïssé class of finite graphs.

We let $\mathbf{R}$ denote the Rado graph.

Ramsey degrees

Fact. For each finitary coloring of the vertices of the Rado graph $\mathbf{R}$, there is a subgraph $\mathbf{R}'$, also a Rado graph, in which the vertices are homogeneous for c.

However, for finite colorings of graphs with more than one vertex, it is not always possible to cut down to one color in a copy of the full Rado graph.

Def. The **Ramsey degree** of a finite graph G is the smallest number t_G such that for each coloring of all copies of G inside $\mathbf{R}$, there is a subgraph $\mathbf{R}'$, also a Rado graph, such that all copies of G in $\mathbf{R}'$ have no more than t_G colors.

The Rado graph has finite Ramsey degrees

Thm. (Sauer) The Rado graph has finite Ramsey degrees for every finite graph.

Key notions in the proof: Let G be a finite graph.

1. Trees can code graphs.
2. There are only finitely many isomorphism types of trees coding G, and each of these can be enveloped uniquely into a finite strong tree.
3. Apply Milliken's Theorem to these 'strong tree envelopes' of the trees coding G.

Coding graphs as subtrees of $2^{<\omega}$

Let G be a graph.

Enumerate the vertices of G in any order as $\langle v_n : n < \omega \rangle$.

The n-th **distinguished node** t_n codes v_n.

For all $i < n$,

$$t_n(\mathrm{lh}(t_i)) = 1 \Leftrightarrow v_n \ E \ v_i.$$

Draw

A graph.
A tree coding this graph.
A diagonal tree coding this graph.
Two types of diagonal trees coding edges.
Their strong tree envelopes.
How the Milliken Theorem is used to prove the finite Ramsey degrees for the Rado graph.

The Universal Triangle-Free Graph

Next, a recent result: The universal triangle-free graph on countably many vertices has finite Ramsey degrees.

Key ideas are

1. Construction of a new almost Ramsey space of 'strong triangle-free trees'.
2. The use of forcing to prove the pigeonhole principle.
3. A new notion of subtree envelope.

The universal homogeneous triangle-free graph

Def. $\mathcal{K}_3$ denotes the Fraïssé class of all countable triangle-free graphs.

H is **universal** for $\mathcal{K}_3$ if H is triangle-free and every countable triangle-free graph embeds into H.

H is **homogeneous** for $\mathcal{K}_3$ if whenver G is a finite triangle-free graph, every embedding of G into H can be extended to an automorphism of H.

Thm. (Henson) There is a countable graph which is universal and homogeneous for $\mathcal{K}_3$.

Any two countable universal homogeneous triangle-free graphs are isomorphic. Let $\mathbf{H}_3$ denote it.

Previous Results

Thm. (Nešetřil/Rödl) $\mathcal{K}_3$ has the Ramsey property:

$$\forall G \leq H \text{ in } \mathcal{K}_3 \ \exists K \in \mathcal{K}_3 \ \ K \to (H)^G_2.$$

Thm. (Komjáth/Rödl) For any finite coloring of the vertices $|\mathbf{H}_3|$, there is an $\mathbf{H} \in \binom{\mathbf{H}_3}{\mathbf{H}_3}$ such that $|\mathbf{H}|$ has one color.

Thm. (Sauer) For any finite coloring of the edges in $\mathbf{H}_3$, there is an $\mathbf{H} \in \binom{\mathbf{H}_3}{\mathbf{H}_3}$ such that the edges in $\mathbf{H}$ take on no more than two colors.

Does $\mathbf{H}_3$ have finite Ramsey degrees?

Question. Given any finite triangle free graph G, is there a number $n_{\mathrm{G}} < \omega$ such that for any coloring c of all copies of G in $\mathbf{H}_3$ into finitely many colors, there is a subgraph $\mathbf{H}$ of $\mathbf{H}_3$ isomorphic to $\mathbf{H}_3$ in which the coloring takes on no more than n_{G} colors?

Constructing copies of $\mathbf{H}_3$

Thm. (Henson). The following property is equivalent to $\mathbf{H}$ being a universal triangle-free graph:

Property (A_3).

(i) $\mathbf{H}$ is triangle-free.

(ii) If A and B are disjoint finite sets of vertices of $\mathbf{H}$, and $\mathbf{H}|A$ (the graph $\mathbf{H}$ restricted to the vertices in A) does not have any edges, then there is another vertex in $\mathbf{H}$ which is connected to every member of A and to no member of B.

Property (A_3) provides a simple way to construct $\mathbf{H}_3$. We use a slight modification in our constructions.

What codes a triangle?

Recall: Trees code graphs.

For $i < j < k$, suppose the vertices $\{v_i, v_j, v_k\}$ are coded by the distinguished nodes t_i, t_j, t_k in $2^{<\omega}$.

The vertices $\{v_i, v_j, v_k\}$ form a triangle if and only if the distinguished nodes t_i, t_j, t_k satisfy $t_k(|t_j|) = t_k(|t_i|) = t_j(|t_i|) = 1$.

Whenever $t_k(|t_i|) = t_j(|t_i|) = 1$, we say that t_k and t_j have **parallel 1's**.

Here $|t|$ is denoting the length of the node t.

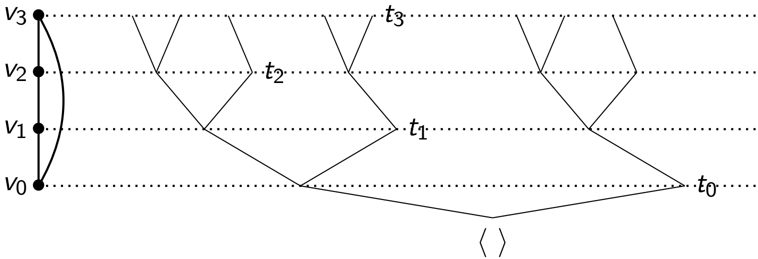
Finite strong triangle-free trees

Finite strong triangle-free trees are trees which code a triangle-free graph and which branch as much as possible, subject to the

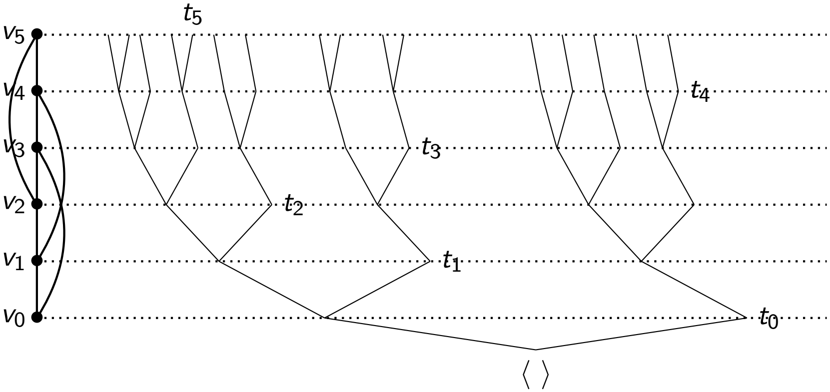
Triangle-Free Extension Criterion: A node t at the level of the n-th distinguished node t_n extends right if and only if t and t_n have no parallel 1's.

Every node always extends left.

Building a strong triangle-free $\mathbb{T}$ to code $\mathbf{H}_3$



Building a strong triangle-free $\mathbb{T}$ to code $\mathbf{H}_3$



Skew strong triangle-free trees

In order to prove **A.4** holds for $r_1[0, T]$, we need to make the trees skew.

The following structure is skew and diagonal, but codes $\mathbf{H}_3$ in exactly the same way that $\mathbb{T}$ does. (Draw)

A node in a strong triangle-free coding tree is a **critical node** if it is either a distinguished node or a splitting node.

The almost Ramsey space $\mathcal{T}(\mathbb{T})$ of strong coding trees

Let $\mathbb{T}$ be a skew strong triangle-free tree densely coding $\mathbf{H}_3$.

Let $\mathcal{T}(\mathbb{T})$ or simply $\mathcal{T}$ denote the collection of all subtrees $T \le \mathbb{T}$ which are isomorphic to $\mathbb{T}$.

Note that each $T \in \mathcal{T}$ codes a copy of $\mathbf{H}_3$.

For subtrees T, T' of $\mathbb{T}$, write $T' \le T$ to denote that T' is a subtree of T and $T' \cong T$.

$r_k(T)$ is the first k levels of T, where the levels are determined by the critical nodes.

The universal triangle-free graph has finite Ramsey degrees

Thm. (D.)

1. There is a skew strong triangle-free tree coding $\mathbb{T}$ coding $\mathbf{H}_3$.
2. $(\mathcal{T}(\mathbb{T}), \leq, r)$ forms a space which satisfies all four of Todocevic's Axioms except **Axiom A.3 (2)**.
3. For each k and each $T \in \mathcal{T}(\mathbb{T})$, $\mathcal{AR}_k|T$ has the Ramsey property, and some stronger statements hold.
4. The universal triangle-free graph has finite Ramsey degrees.

Finite Ramsey degrees in $\mathbf{H}_3$

The rest of the proof of finite Ramsey degrees follows these steps.

1. Let G be a finite triangle-free graph.
2. There are only finitely many ways to code G by a diagonal tree.
3. Define a new kind of subtree envelope for the finite collection of all diagonal trees with distinguished nodes coding G.
4. Thin to a tree $\mathbb{S}$ in which the finitely many embedding types are homogenized. This uses an extended version of theorem that $\mathcal{AR}_k$ is Ramsey.
5. After we've finished homogenizing $\mathbb{S}$ for the finitely many triangle graphs G_i, thin $\mathbb{S}$ to a diagonal subtree $\mathbb{D}$ coding $\mathbf{H}_3$ and a set of auxiliary witnessing nodes W from $\mathbb{S}$ so that for each tree coding a tree in our collection of types there are nodes in W available to get an envelope in $\mathbb{S}$.

A.4 and stronger statements are proved by forcing

Thm. (D.) Let $T \in \mathcal{T}$, $k < \omega$, $c : r_{k+1}[r_k(T), T] \to 2$ be given. Then there is an $S \in [r_k(T), T]$ such that c is constant on $r_{k+1}[r_k(T), S]$.

The proof builds on Harrington's forcing proof of Halpern-Läuchli Theorem.

The difficulties were

1. If we don't use skew trees, we cannot obtain **A.4** for $r_1[0, T]$. So we had to use skew trees in the forcing.
2. How to define a forcing for skew trees with distinguished nodes which is transitive, does not add triangles, but has properties similar to Cohen forcing in order to extend to homogeneous level sets?

Proof of **A.4**

Pf. Let $c : r_{k+1}[k, T] \to 2$.

Let $A = \{\text{immediate successors of } r_k(T) \text{ in } T\}$ and $d + 1 = |A|$.

List the nodes of A as $s_0, \dots, s_d$, where s_d is the node of A that the critical node in each member of $r_{k+1}[A, T]$ must extend.

Let L denote the set of levels l of T such that there is a member of $r_{k+1}[A, T]$ with critical node at level l.

Note that $L = \{l : \text{the distinguished node in } T(l) \text{ extends } s_d\}$.

For each for $i \leq d$, let $T_i = \{t \in T : t \supseteq s_i\}$.

Let $\kappa = \beth_{2d}$. The following forcing notion $\mathbb{P}$ will add κ many generic paths $\dot{b}_{i,\alpha}$ through each T_i, $i < d$, and one path $\dot{b}_d$ through T_d.

$$p \in \mathbb{P} \text{ iff is a function of the form } p : \{d\} \cup (d \times \vec{\delta}_p) \to T(l_p),$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, satisfying

(i) $p(d)$ is the critical node of $T_d(l_p)$.

(ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i(l_p)$.

Def. $\{t_i : i \le d\}$ satisfies $(*)$ over A iff each $t_i \supseteq s_i$, t_d is a critical node, and

1. If s_d is extended to a distinguished node, then $A \cup \{t_i : i \le d\}$ satisfies the Parallel 1's Criterion; and
2. If s_d is extended to a splitting node, then all parallel 1's in $A \cup \{t_i : i \le d\}$ is witnessed by a distinguished node in A.

$q \le p$ if and only if either

1. $l_q = l_p$ and $q \supseteq p$; or else
2. $l_q > l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, $q(d) \supset p(d)$, and
   (a) For each $\delta \in \vec{\delta}_p$ and $i < d$, $q(i, \delta) \supset p(i, \delta)$,
   (b) For each increasing sequence $(\alpha_0, \dots, \alpha_d) \in (\vec{\delta}_p)^d$ which $\{p(i, \alpha_i) : i < d\} \cup \{p(d)\}$ satisfying $(*)$ over A, then $\{q(i, \alpha_i) : i < d\} \cup \{q(d)\}$ also satisfies $(*)$ over A.

Let $\dot{L}_d$ be a name for the set of levels of distinguished nodes in $\dot{b}_d$ and $\dot{\mathcal{U}}$ be a name for an ultrafilter on $\dot{L}_d$.

For each $\vec{\alpha} \in [\kappa]^d$, choose a condition $p_{\vec{\alpha}} \in \mathbb{P}$ such that

1. $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$,
2. $\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p(d)\}$ satisfies $(*)$ over A.
3. $p_{\vec{\alpha}} \Vdash$ "There is an $\varepsilon \in 2$ such that $c(\dot{b}_{\vec{\alpha}}(l)) = \varepsilon$ for $\dot{\mathcal{U}}$ many l",
4. $p_{\vec{\alpha}}$ decides a value for ε, call it $\varepsilon_{\vec{\alpha}}$.
5. $\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p(d)\}$ takes value $\varepsilon_{\vec{\alpha}}$.

Make a coloring f on $[\kappa]^{2d}$ which codes all the information we need.

Let $\mathcal{I}$ denote the collection of all functions $\iota : 2d \to 2d$ such that $\iota \restriction \{0, 2, \ldots, 2d-2\}$ and $\iota \restriction \{1, 3, \ldots, 2d-1\}$ are strictly increasing sequences and $\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}$.

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, letting $\vec{\alpha}$ denote $\iota_e(\vec{\theta})$ and $\vec{\beta}$ denote $\iota_o(\vec{\theta})$, let

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle & \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, , \langle \langle i, j \rangle : i < d,\ j < k_{\vec{\alpha}},\ \text{and}\ \vec{\delta}_{\vec{\alpha}}(j) = \alpha_i \rangle, \\ & \langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle, \langle \langle p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) : j < k_{\vec{\beta}} \rangle : i < d \rangle, \\ & \langle \langle j, k \rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle. \end{aligned} \tag{2}$$

Let $f(\vec{\theta})$ be the sequence $\langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$, where $\mathcal{I}$ is given some fixed ordering.

By the Erdős-Rado Theorem, there is $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Claim. There are $K_0 < \cdots < K_{d-1}$ infinite such that the set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

Moreover, there are nodes t_i^*, $i < d$, such that for each $(\alpha_i : i < d) \in \prod_{i<d} K_i$, $p_{\vec{\alpha}}(i, \alpha_i) = t_i^*$. Further, all $p_{\vec{\alpha}}(d) = t_d^*$.

Then extend these t_i^*, $i \leq d$, using $K_0 < \cdots < K_{d-1}$ to build a member $S \in [r_k(T), T]$ such that c is constant on $r_{k+1}[r_k(T), S]$.

We conclude this tutorial with some Ramsey results for measurable cardinals.

Extension of the Halpern-Läuchli Theorem to a measurable

Thm. (Shelah) Suppose V is a model of ZFC with a measurable cardinal κ such that for λ large enough, after forcing with $\mathrm{Add}(\kappa, \lambda)$, κ remains measurable.

Then for each $1 \leq m < \omega$ and any coloring of the m-sized level sets of the tree $2^{<\kappa}$ into less than κ colors, there is a strong subtree $T \subseteq 2^{<\kappa}$ on which the coloring takes only finitely many colors.

In fact, the color depends exactly on the type of the induced subtree below the m-many nodes.

Thm. (Džamonja/Larson/Mitchell) Suppose V is a model of ZFC with a measurable cardinal κ such that for λ large enough, after forcing with $\mathrm{Add}(\kappa, \lambda)$, κ remains measurable.

Then for each $1 \leq m < \omega$ and any coloring of the m-sized antichains in the tree $2^{<\kappa}$ into less than κ colors, there is a strong subtree $T \subseteq 2^{<\kappa}$ on which the coloring takes only finitely many colors.

(Again, the color depends on the type of the induced subtree below the m-many nodes.)

This is applied in two papers of Džamonja, Larson and Mitchell to prove that the Rado graph on κ many vertices has finite Ramsey degrees, and that colorings of the m-sized subsets of the κ-rationals has finite Ramsey degrees.

Extensions to more than one tree and wider trees

A tree $T \subseteq {}^{<\kappa}\kappa$ is a κ-**tree** if T has cardinality κ and every level of T has cardinality less than κ.

$T \subseteq {}^{<\kappa}\kappa$ is **regular** if it is a perfect κ-tree in which every maximal branch has cofinality κ.

For $\zeta < \kappa$, let $T(\zeta) = T \cap {}^{\zeta}\kappa$.

Let $T \subseteq {}^{<\kappa}\kappa$ be regular. A tree $S \subseteq T$ is a **strong subtree** of T if S is regular and there is some $A \subseteq \kappa$ cofinal in κ such that for each $s \in S$,

1. s splits iff t has length $\zeta \in A$, and
2. For each $\zeta \in A$ and $s \in S(\zeta)$, s is maximally branching in T.

HL(δ, σ, κ)

Def. Let $\delta, \sigma > 0$ be ordinals and κ be an infinite cardinal.

HL(δ, σ, κ) is the following statement:

Given any sequence $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ of regular trees and a coloring

$$c : \bigcup_{\zeta<\kappa} \prod_{i<\delta} T_i(\zeta) \to \sigma,$$

there exists a sequence of trees $\langle S_i : i < \delta \rangle$ and $A \in [\kappa]^\kappa$ such that

1. each S_i is a strong subtree of T_i as witnessed by $A \subseteq \kappa$, and
2. there is some $\sigma' < \sigma$ such that c has color σ' on $\bigcup_{\zeta \in A} \prod_{i<\delta} S_i(\zeta)$.

Def. A cardinal κ is **$\kappa + d$-strong** if there is an elementary embedding $j : V \to M$ with critical point κ such that $V_{\kappa+d} = M_{\kappa+d}$.

Thm. (D./Hathaway) Let $d \geq 1$ be any finite integer and suppose that κ is a $\kappa + d$-strong cardinal in a model V of ZFC satisfying GCH. Then there is a forcing extension in which κ remains measurable and $\mathrm{HL}(d, \sigma, \kappa)$ holds, for all $\sigma < \kappa$.

Open Problem. Find the exact consistency strength of $\mathrm{HL}(d, \sigma, \kappa)$ for κ a measurable cardinal.

Conclusion

1. Ramsey theory and its development is useful for better understanding ultrafilters forced by σ-closed forcings.
2. Forcing can be used to prove new Ramsey theorems in ZFC.
3. The interaction between these two fields enriches both.
4. There are many related open problems in both directions desiring your work.

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