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The hyper-weak distributive law and a related game in Boolean algebras

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Abstract

We discuss the relationship between various weak distributive laws and games in Boolean algebras. In the first part we give some game characterizations for certain forms of Prikry's "hyper-weak distributive laws", and in the second part we construct Suslin algebras in which neither player wins a certain hyper-weak distributivity game. We conclude that in the constructible universe L, all the distributivity games considered in this paper may be undetermined.

Keywords: Boolean algebra; Distributive law; Game; Suslin tree

1. Introduction

We recall that if $\mathbb{B}$ is a Boolean algebra then it satisfies the *weak (κ, λ)-distributive law* if for each family $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda}$ such that $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta}$ exists, and $\bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}$ exists for all $f : \kappa \to [\lambda]^{<\omega}$, then

$$\bigvee_{f:\kappa\to[\lambda]^{<\omega}} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}$$

also exists and

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} = \bigvee_{f:\kappa\to[\lambda]^{<\omega}} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}.$$

Notice that

(1) We are not insisting that $\mathbb{B}$ be complete.

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(2) In general this law can hold in a Boolean algebra without holding in its completion.
(3) When $\mathbb{B}$ is complete this distributive law has a natural statement in terms of forcing. Namely it asserts that for every $\mathbb{B}$-name $\dot{g}$ for a function from κ to λ, $\Vdash$ "there is a function $f \in V$ from κ to $[\lambda]^{<\omega}$ such that $\forall \alpha\ \dot{g}(\alpha) \in \check{f}(\alpha)$".
(4) We are following the conventions of Koppelberg's handbook [1]. Other definitions of the term "*weak* (κ, λ)-*distributive law*" appear in the literature, for example Namba [2] uses this term for a version where λ appears in place of ω.

The hyper-weak (ω, ω)-distributive law (defined in the next section) was formulated by Prikry as a generalization of the weak (ω, ω)-distributive law. His motivation was a problem of von Neumann, whether it is consistent with ZFC that the countable chain condition and the weak (ω, ω)-distributive law completely characterize measurable Boolean algebras among Boolean σ-algebras [3]. Consistent counter-examples to von Neumann's proposed characterization of measurable algebras were obtained by Maharam [4], Jensen [5], Glówczyński [6], and Veličković [7]. For example a Suslin algebra is c.c.c. and (ω, ∞)-distributive, whereas a measurable algebra adds new reals.

Maharam [4] identified a class of Boolean algebras (the *Maharam algebras*) such that every measurable algebra is a Maharam algebra and every Maharam algebra is c.c.c. and weakly (ω, ω)-distributive. The notable *Control Measure Problem* asked whether every Maharam algebra is measurable; Talagrand [8] recently showed that this is false in ZFC. It is also known that consistently every c.c.c. and weakly (ω, ω)-distributive algebra is Maharam [9,10].

Prikry's idea was to try to find in ZFC a complete, non-measurable Boolean algebra satisfying the countable chain condition (c.c.c.) in which some weaker form of the weak (ω, ω)-distributive law holds. This would give a type of lower bound on von Neumann's problem within ZFC. Specifically, Prikry asked the following question.

Open Problem 1 (*Prikry*). Can one find in ZFC a complete c.c.c. Boolean algebra in which the hyper-weak (ω, ω)-distributive law holds, but the weak (ω, ω)-distributive law fails everywhere?

We shall call such a Boolean algebra a *P-algebra*. Finding a P-algebra in ZFC turns out to be harder than it might seem at first glance. A forcing poset $\mathbb{P}$ whose conditions can be coded as reals is said to be *Suslin* if $\mathbb{P}$ and the ordering and incompatibility relations on $\mathbb{P}$ are all $\boldsymbol{\Sigma}^1_1$ (in the codes); most of the well known forcing posets for adding new reals are of this type. No Boolean algebra in which a c.c.c. Suslin forcing embeds as a dense subset can be a P-algebra, for Shelah has shown that for each c.c.c. Suslin forcing $\mathbb{P}$, the weak (ω, ω)-distributive law holds in r.o.$(\mathbb{P})$ iff $\mathbb{P}$ does not add a Cohen real [11].

Since the hyper-weak (ω, ω)-distributive law is weaker than the weak (ω, ω)-distributive law and implies that no Cohen reals are added, Shelah's result implies every c.c.c. Suslin forcing which is hyper-weakly (ω, ω)-distributive is also weakly (ω, ω)-distributive. In [12], Dobrinen investigated two families of Suslin c.c.c. and one family of non-Suslin c.c.c. forcings which give rise to non-measurable Boolean algebras, and found that each of these algebras adds a Cohen real. Further, Błaszczyk and Shelah have shown that the Cohen algebra embeds into each σ-centered complete Boolean algebra if and only if there are no nowhere dense ultrafilters over ω [13]. Since Shelah has shown that the existence of nowhere dense ultrafilters is independent of ZFC [14], such Boolean algebras cannot be shown to be P-algebras in ZFC. At least the existence of a P-algebra is consistent with ZFC. Recall the version of Mathias forcing where the conditions are pairs (s, A) with $s \in [\omega]^{<\omega}$ and $A \in U$ for a fixed Ramsey ultrafilter U. Simpson [15] pointed out that the regular open algebra of this forcing poset is a c.c.c., hyper-weakly (ω, ω)-distributive algebra in which the weak (ω, ω)-distributive law fails everywhere.

Although the problem of finding a P-algebra in ZFC remains open, the hyper-weak (ω, ω)-distributive law and its generalizations for larger cardinals have proved useful in the realm of games. Let us give a bit of background into the connections between distributive laws and games in Boolean algebras.

Jech investigated various distributive laws and related games [16]. Among other things, he gave a game-theoretic characterization of the (ω, λ)-distributive law. Dobrinen extended this to more general distributive laws in [17]. Kamburelis solved an open problem from [16] giving a best possible result connecting the weak (ω, λ)-distributive law and its related game [18]. This was more recently generalized by Dobrinen in [19]. In Section 3 of this paper, we give connections between the hyper-weak (κ, λ)-distributive law and a related game, obtaining a game-theoretic characterization of the hyper-weak (κ, λ)-distributive law for many pairs of cardinals κ, λ, under GCH.

When we associate games with distributive laws, the property "II has a winning strategy" implies that the related distributive law holds. In [16], Jech used $\diamondsuit$ to construct a Suslin algebra in which the game related to the $(\omega, 2)$-distributive law is undetermined. Dobrinen generalized that result to κ^+-Suslin algebras for κ regular uncountable

[17,20]. In Section 4, we improve on this result in several ways. For every infinite cardinal κ and each infinite regular cardinal $\nu \leq \mathrm{cf}(\kappa)$, we use $\square_\kappa$, $\diamondsuit_{\kappa^+}(S)$ for all stationary subsets $S \subseteq \mathrm{cof}(\nu)$, and $\kappa^{<\nu} = \kappa$ to construct a κ^+-Suslin algebra which contains a $< \nu$-closed dense subset and in which II does not have a winning strategy for the game related to the hyper-weak (ν, κ)-distributive law. As we shall show, it follows that in L, for any distributive law mentioned in this paper, there is a cardinal κ and a κ^+-Suslin algebra on which the related game is undetermined.

Notation 1. *We use the following notation from [1]: Given cardinals κ, λ, μ, $\mathrm{Fn}(\kappa, \lambda, \mu)$ denotes the set of partial functions from κ to λ of cardinality $< \mu$. Given a separative partial ordering $\mathbb{P}$, let $\mathrm{r.o.}(\mathbb{P})$ denote the Boolean completion of $\mathbb{P}$; that is, the complete Boolean algebra containing $\mathbb{P}$ as a dense subset. $e : \mathbb{P} \to \mathrm{r.o.}(\mathbb{P})$ denotes the canonical embedding of $\mathbb{P}$ into $\mathrm{r.o.}(\mathbb{P})$, the "regular open algebra" on $\mathbb{P}$.*

2. Generalized distributive laws: Definitions and basic facts

We start by reviewing the three-parameter distributive law, which subsumes all the conventional generalized distributive laws. Throughout this paper, we let $\mathbb{B}$ denote a Boolean algebra and $\mathbb{B}^+$ denote $\mathbb{B} \setminus \{\mathbf{0}\}$. The following is a generalization of the standard definition of the (κ, λ)-distributive law in [1] to three parameters.

Definition 2. If κ, λ, μ are cardinals with $2 \leq \mu \leq \lambda$, a Boolean algebra $\mathbb{B}$ satisfies the $(\kappa, \lambda, < \mu)$-*distributive law* if for each family $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda}$ such that $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta}$ exists, and $\bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}$ exists for all $f : \kappa \to [\lambda]^{<\mu}$, then $\bigvee_{f:\kappa\to[\lambda]^{<\mu}} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}$ exists and

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} = \bigvee_{f:\kappa\to[\lambda]^{<\mu}} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}. \tag{1}$$

$\mathbb{B}$ satisfies the $(\kappa, \infty, < \mu)$-*distributive law* if the $(\kappa, \lambda, < \mu)$-distributive law holds in $\mathbb{B}$ for all λ. We say that the $(\kappa, \lambda, < \mu)$-distributive law *fails everywhere* in $\mathbb{B}$ if there exists a family $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda} \subseteq \mathbb{B}$ such that $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} = \mathbf{1}$ and $\bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta} = \mathbf{0}$ for all $f : \kappa \to [\lambda]^{<\mu}$.

Remark 3. The $(\kappa, \lambda, < 2)$-distributive law is the familiar (κ, λ)-distributive law, and the $(\kappa, \lambda, < \omega)$-distributive law is the weak (κ, λ)-distributive law discussed in Section 1. Saying that "the $(\kappa, \lambda, < \mu)$-distributive law fails everywhere in $\mathbb{B}$" is equivalent to saying that "$\mathbb{B}$ is $(\kappa, \lambda, < \mu)$-nowhere distributive" in Koppelberg's terminology [1].

Prikry formulated the following weakening of the $(\kappa, \lambda, < \mu)$-distributive law.

Definition 4 (*Prikry [21]*). For κ, λ cardinals with $\lambda \geq \omega$, a Boolean algebra $\mathbb{B}$ satisfies the *hyper-weak* (κ, λ)-*distributive law* if for each family $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda}$ of elements of $\mathbb{B}$, if $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta}$ exists and $\bigwedge_{\alpha<\kappa} \bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} b_{\alpha\beta}$ exists for all $f : \kappa \to \lambda$, then $\bigvee_{f:\kappa\to\lambda} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} b_{\alpha\beta}$ exists and

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} = \bigvee_{f:\kappa\to\lambda} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} b_{\alpha\beta}. \tag{2}$$

$\mathbb{B}$ satisfies the *hyper-weak* (κ, ∞)-*distributive law* if the hyper-weak (κ, λ)-distributive law holds in $\mathbb{B}$ for all $\lambda \geq \omega$. We say that the hyper-weak (κ, λ)-distributive law *fails everywhere in* $\mathbb{B}$ if there exists $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda} \subseteq \mathbb{B}$ such that $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} = \mathbf{1}$ and $\bigwedge_{\alpha<\kappa} \bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} b_{\alpha\beta} = \mathbf{0}$ for all $f : \kappa \to \lambda$.

The next fact follows naturally from the definitions.

Fact 5. *For each Boolean algebra $\mathbb{B}$, the following hold.*

(1) *For all cardinals $\kappa_0 \leq \kappa_1$, $2 \leq \mu_0 \leq \mu_1$, and $\lambda_0 \leq \lambda_1$, if $\mathbb{B}$ satisfies the $(\kappa_1, \lambda_1, < \mu_0)$-distributive law, then $\mathbb{B}$ satisfies the $(\kappa_0, \lambda_0, < \mu_1)$-distributive law.*
(2) *For all cardinals $\kappa_0 \leq \kappa_1$ and $\omega \leq \lambda_0 \leq \lambda_1$, if $\mathbb{B}$ satisfies the hyper-weak (κ_1, λ_0)-distributive law, then $\mathbb{B}$ satisfies the hyper-weak (κ_0, λ_1)-distributive law. Hence, the hyper-weak (κ, ∞)-distributive law is equivalent to the hyper-weak (κ, ω)-distributive law.*
(3) *For all cardinals $\kappa, \lambda, \mu, \nu$ with $2 \leq \mu \leq \lambda$ and $\nu \geq \max(\omega, \mu)$, if $\mathbb{B}$ satisfies the $(\kappa, \lambda, < \mu)$-distributive law, then $\mathbb{B}$ satisfies the hyper-weak (κ, ν)-distributive law.*

Examples 6. (1) If κ, λ are cardinals with $\lambda \geq \omega$, the hyper-weak (κ, λ)-distributive law holds in each Boolean algebra satisfying the λ-chain condition.

(2) Let κ, λ, μ be cardinals with κ, μ regular and $\omega \leq \mu \leq \kappa$, and let $\nu = \max(\lambda, \mu)$. R.o.(Fn(κ, λ, μ)) satisfies the (ρ, ∞)-distributive law for each $\rho < \mu$, but the hyper-weak (μ, ν)-distributive law fails everywhere. In particular, when $\mu = \kappa$, for each $\rho < \kappa$, the (ρ, ∞)-distributive law holds in r.o.(Fn$(\kappa, \lambda, \kappa)$), but the hyper-weak (κ, ν)-distributive law fails everywhere in r.o.(Fn$(\kappa, \lambda, \kappa)$).

(3) In each free Boolean algebra on infinitely many generators, for each cardinal $\kappa \geq \omega$, the hyper-weak (κ, ω)-distributive law fails everywhere, but the hyper-weak (κ, ω_1)-distributive law holds.

(4) In Laver, Mathias, and Miller forcings, the hyper-weak (ω, ω)-distributive law holds, but the weak (ω, ω)-distributive law fails everywhere.

Definition 7 (*[1]*). For any cardinal κ and $b \in \mathbb{B}^+$, a collection $\{b_\alpha : \alpha < \kappa\} \subseteq \mathbb{B}$ is a *quasi-partition of b* if each $b_\alpha \leq b$, $\{b_\alpha : \alpha < \kappa\}$ is pairwise disjoint, and $\bigvee_{\alpha<\kappa} b_\alpha = b$. $\{b_\alpha : \alpha < \kappa\} \subseteq \mathbb{B}$ is a *partition of b* if it is a quasi-partition of b where each $b_\alpha > \mathbf{0}$.

In a λ-complete Boolean algebra $\mathbb{B}$, whether the hyper-weak (κ, λ)-distributive law holds in $\mathbb{B}$ can be determined by looking only at quasi-partitions or partitions of unity.

Fact 8. *For all cardinals κ, λ with $\lambda \geq \omega$, for each λ-complete Boolean algebra $\mathbb{B}$, the following are equivalent.*

(1) *The hyper-weak (κ, λ)-distributive law holds in $\mathbb{B}$.*
(2) *For all $b \in \mathbb{B}^+$, Eq. (2) of Definition 4 holds for all families $\{b_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, of (quasi-)partitions of b in $\mathbb{B}$.*
(3) *Eq. (2) of Definition 4 holds for all families $\{b_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, of (quasi-)partitions of unity in $\mathbb{B}$.*

Remark 9. It is well known that if the hyper-weak (κ, λ)-distributive law fails then there is $b \in \mathbb{B}^+$ such that it fails everywhere in $\mathbb{B} \restriction b$.

To see this suppose that there is a family $\{b_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, of quasi-partitions of unity for which the hyper-weak (κ, λ)-distributive law fails. Then there is a $c < \mathbf{1}$ such that for each $f : \kappa \to \lambda$, $c \geq \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} b_{\alpha\beta}$. Let $a = \mathbf{1} \setminus c$ and let $a_{\alpha\beta} = a \wedge b_{\alpha\beta}$. Then each $\{a_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, is a quasi-partition of a, and $\bigvee_{f:\kappa\to\lambda} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} a_{\alpha\beta} = \mathbf{0}$. We can make these into partitions of a as follows: Given $\alpha < \kappa$, let $B_\alpha = \{\beta < \lambda : a_{\alpha\beta} > \mathbf{0}\}$. If $|B_\alpha| < \lambda$, then $\{a_{\alpha\beta} : \beta < \lambda\}$ does not contribute to the failure of the hyper-weak(κ, λ)-distributive law. Let $K = \{\alpha < \kappa : |B_\alpha| = \lambda\}$. For each $\alpha \in K$, $\{a_{\alpha\beta} : \beta \in B_\alpha\}$ is a partition of a, and $\bigvee_{f:K\to\lambda} \bigwedge_{\alpha\in K} \bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} a_{\alpha\beta} = \mathbf{0}$.

The following is a characterization of the hyper-weak (κ, λ)-distributive law for a partial ordering $(\mathbb{P}, \leq)$ via its Boolean completion r.o.$(\mathbb{P})$. It holds whether or not $\mathbb{P}$ is separative.

Fact 10. *Given a partial ordering $(\mathbb{P}, \leq)$, the following are equivalent.*

(1) *The hyper-weak (κ, λ)-distributive law holds in* r.o.$(\mathbb{P})$.
(2) *If $\mathcal{W}_\alpha = \{P_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, is a family such that for each $\alpha < \kappa$,*
  (a) $\beta \neq \beta' \longrightarrow P_{\alpha\beta} \cap P_{\alpha\beta'} = \emptyset$,
  (b) $\bigcup_{\beta<\lambda} P_{\alpha\beta}$ *is a maximal antichain in $\mathbb{P}$,*
  then there exists a maximal antichain $\mathcal{Q} \subseteq \mathbb{P}$ such that $\forall q \in \mathcal{Q}$, $\forall \alpha < \kappa$, $\exists \beta < \lambda$ such that $\forall p \in P_{\alpha\beta}$, p and q are incompatible.

Remark 11. Prikry observed that taking suprema over all but one element of λ on the right hand side of (2) in Definition 4 of the hyper-weak (κ, λ)-distributive law is equivalent to taking suprema over subsets of λ whose complements have cardinality λ; that is, replacing the right hand side of (2) with $\bigvee_{f:\kappa\to\mathcal{S}} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}$, where $\mathcal{S} = \{X \subseteq \lambda : |\lambda \setminus X| = \lambda\}$. (See [22].)

Let $\mathbb{B}$ be complete. We let α, β denote ordinals in V and κ, λ denote cardinals in V. For $x \in V$ $\check{x}$ denotes the canonical $\mathbb{B}$-name for x. We use $\dot{x}$ to denote general $\mathbb{B}$-names. The following is an easy forcing equivalent of general distributive laws.

Fact 12 (*Folklore*). *Let $\mathbb{B}$ be complete. The $(\kappa, \lambda, < \mu)$-distributive law holds in $\mathbb{B}$ iff for each $\mathbb{B}$-name $\dot{g}$ for a function from $\check{\kappa}$ to $\check{\lambda}$, $\Vdash$ "there is $f \in V$ such that $f : \kappa \to [\lambda]^{<\mu}$ and $\forall \alpha < \check{\kappa}$ $\dot{g}(\alpha) \in \check{f}(\alpha)$".*

In particular, a complete Boolean algebra $\mathbb{B}$ satisfies the (κ, λ)-distributive law iff forcing with $\mathbb{B}^+$ adds no new functions from $\check{\kappa}$ to $\check{\lambda}$. The following is the analog for the hyper-weak (κ, λ)-distributive law, and is proved in the same way.

Fact 13. *Let $\mathbb{B}$ be complete. The hyper-weak (κ, λ)-distributive law holds in $\mathbb{B}$ iff for each $\mathbb{B}$-name $\dot{g}$ for a function from $\check{\kappa}$ to $\check{\lambda}$, $\Vdash$ "there is $f \in V$ such that $f : \kappa \to \lambda$ and $\forall \alpha < \check{\kappa}\ \dot{g}(\alpha) \neq \check{f}(\alpha)$".*

3. A game related to the hyper-weak distributive law

We begin by reviewing the following game investigated in [17], which generalizes a game of Jech in [16]. This game is related to the $(\kappa, \lambda, < \mu)$-distributive law. (See Theorem 16(1) below.)

Definition 14 (*[17]*). Given cardinals κ, λ, μ with $2 \le \mu \le \lambda$, the game $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ is played between two players in a $\max(\kappa^+, \mu)$-complete Boolean algebra $\mathbb{B}$ as follows: At the beginning of the game, player I fixes some $a \in \mathbb{B}^+$. For $\alpha < \kappa$, the α-th round is played as follows: player I chooses a partition W_α of a such that $|W_\alpha| \le \lambda$; then player II chooses some $E_\alpha \in [W_\alpha]^{<\mu}$. In this manner, the two players construct a sequence of length κ

$$\langle a, W_0, E_0, W_1, E_1, \ldots, W_\alpha, E_\alpha, \ldots : \alpha < \kappa\rangle \tag{3}$$

called a *play of the game*. I wins the play (3) if

$$\bigwedge_{\alpha<\kappa} \bigvee E_\alpha = \mathbf{0}, \tag{4}$$

and II wins otherwise.

$\mathcal{G}^{\kappa}_{<2}(\lambda)$ is usually denoted as $\mathcal{G}^{\kappa}_{1}(\lambda)$. $\mathcal{G}^{\kappa}_{<\mu}(\infty)$ is the game played just as $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$, except now player I can choose partitions of any size.

Remark 15. If $\mathbb{B}$ has a $< \nu$-closed dense subset, then for each $\rho < \nu$, player II has a winning strategy for $\mathcal{G}^{\rho}_{1}(\infty)$; moreover, II even has a winning strategy for the harder game $G^{\mathrm{I}}_{\rho^+}$ invented by Foreman (see [23]).

The following was proved by Dobrinen [17,20].

Theorem 16 (*Dobrinen [17,20]*). (1) *For a* $\max(\kappa^+, \mu)$*-complete Boolean algebra $\mathbb{B}$, the $(\kappa, \lambda, < \mu)$-distributive law fails in $\mathbb{B}$ $\Longrightarrow$ I has a winning strategy for $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ in $\mathbb{B}$ $\Longrightarrow$ the $((\lambda^{<\mu})^{<\kappa}, \lambda, < \mu)$-distributive law fails in $\mathbb{B}$.*
(2) $\kappa^{<\kappa} = \kappa$ *and* $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$ $\Longrightarrow$ *there is a κ^+-Suslin algebra which has a $< \kappa$-closed dense subset and in which neither player has a winning strategy for $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ for all λ, μ with $2 \le \mu \le \min(\lambda, \kappa)$.*

The following game corresponds in a natural way to the hyper-weak (κ, λ)-distributive law. (See Fact 18 and Theorems 28 and 29 below.)

Definition 17 (*Dobrinen–Prikry [22]*). Given cardinals κ, λ with $\lambda \ge \omega$, the game $\mathcal{G}^{\kappa}_{\lambda-1}$ is played between two players in a κ^+-complete Boolean algebra $\mathbb{B}$ as follows: At the beginning of the game, player I fixes some $a \in \mathbb{B}^+$. For $\alpha < \kappa$, the α-th round is played as follows: player I chooses a quasi-partition W_α of a such that $|W_\alpha| = \lambda$; then player II chooses one $b_\alpha \in W_\alpha$ to "leave out" and "plays" $\bigvee(W_\alpha \setminus \{b_\alpha\})$, equivalently $a \setminus b_\alpha$. In this manner, the two players construct a sequence of length κ

$$\langle a, W_0, b_0, W_1, b_1, \ldots, W_\alpha, b_\alpha, \ldots : \alpha < \kappa\rangle \tag{5}$$

called a *play of the game*. I *wins the play* (5) if

$$\bigwedge_{\alpha<\kappa} a \setminus b_\alpha = \mathbf{0}, \tag{6}$$

which happens if and only if $\bigvee_{\alpha<\kappa} b_\alpha = a$, otherwise II wins. This game is named $\mathcal{G}^{\kappa}_{\lambda-1}$, because II plays "all but one" piece, or "λ minus 1-many" pieces, from each quasi-partition.

Fact 18. *For each* $\max(\kappa^+, \lambda)$*-complete Boolean algebra $\mathbb{B}$, if II has a winning strategy for $\mathcal{G}^{\kappa}_{\lambda-1}$ in $\mathbb{B}$, then $\mathbb{B}$ satisfies the hyper-weak (κ, λ)-distributive law.*

Remark 19. Note that in the game $\mathcal{G}^{\kappa}_{\lambda-1}$, player I is required to choose quasi-partitions of size exactly λ. If not, then I would always choose partitions of size ω, since this maximizes I's chances of winning.

The following fact relates the various games.

Fact 20. *For all $\kappa_0 \le \kappa_1$, $2 \le \mu_0 \le \mu_1 \le \lambda_0 \le \lambda_1 \le \lambda_2$, and $\omega \le \lambda_1$, the following diagram shows the implications for the existence of a winning strategy for the two players.*

$$\begin{array}{ccc} \mathcal{G}^{\kappa_1}_{<\mu_0}(\lambda_1) & \begin{array}{c}\xrightarrow{\text{II}} \\ \xleftarrow[\text{I}]{}\end{array} & \mathcal{G}^{\kappa_1}_{\lambda_1-1} \\ \text{I}\uparrow\ \downarrow\text{II} & & \text{I}\uparrow\ \downarrow\text{II} \\ \mathcal{G}^{\kappa_0}_{<\mu_1}(\lambda_0) & \begin{array}{c}\xrightarrow{\text{II}} \\ \xleftarrow[\text{I}]{}\end{array} & \mathcal{G}^{\kappa_0}_{\lambda_2-1} \end{array} \tag{7}$$

For example, $\mathcal{G}^{\kappa_1}_{<\mu_0}(\lambda_1) \xrightarrow{\text{II}} \mathcal{G}^{\kappa_1}_{\lambda_1-1}$ means that for each Boolean algebra $\mathbb{B}$ in which both games are defined, if II has a winning strategy for $\mathcal{G}^{\kappa_1}_{<\mu_0}(\lambda_1)$ in $\mathbb{B}$, then II also has a winning strategy for $\mathcal{G}^{\kappa_1}_{\lambda_1-1}$ in $\mathbb{B}$.

Examples 21. (1) For cardinals κ, λ with $\lambda \ge \omega$, if $\mathbb{B}$ satisfies the λ-chain condition, then II wins $\mathcal{G}^{\kappa}_{\lambda-1}$ in $\mathbb{B}$.

(2) Let κ, λ, μ be cardinals with κ, μ regular and $\omega \le \mu \le \kappa$, and let $\nu = \max(\lambda, \mu)$. $\mathrm{Fn}(\kappa, \lambda, \mu)$ is $< \mu$-closed; so for every $\rho < \mu$, II has a winning strategy for $\mathcal{G}^{\rho}_{1}(\infty)$ in r.o.$(\mathrm{Fn}(\kappa, \lambda, \mu))$. However, I has a winning strategy for $\mathcal{G}^{\kappa}_{\nu-1}$ in r.o.$(\mathrm{Fn}(\kappa, \lambda, \mu))$. In particular, when $\mu = \kappa$, for every $\rho < \kappa$, II wins $\mathcal{G}^{\rho}_{1}(\infty)$, but I wins $\mathcal{G}^{\kappa}_{\nu-1}$ in r.o.$(\mathrm{Fn}(\kappa, \lambda, \kappa))$.

(3) In each free Boolean algebra on infinitely many generators, for every cardinal $\kappa \ge \omega$, I wins $\mathcal{G}^{\kappa}_{\omega-1}$ but II wins $\mathcal{G}^{\kappa}_{\omega_1-1}$.

(4) In Laver, Mathias, and Miller forcings, II wins $\mathcal{G}^{\omega}_{\omega-1}$, but I wins $\mathcal{G}^{\omega}_{\mathrm{fin}}(\omega)$.

Laver, Mathias, and Miller forcings are specific cases of a more general class of forcings $\mathbb{P}$ in which II has a winning strategy for $\mathcal{G}^{\omega}_{\omega-1}$ in r.o.$(\mathbb{P})$. We review the following definitions.

Definition 22 (*[24]*). A partial ordering $(\mathbb{P}, \le)$ satisfies *Axiom A* if there exists a sequence of partial orderings $\le_n$, $n < \omega$, on $\mathbb{P}$ satisfying the following:

(1) $\le_0$ is $\le$, and for all $n < \omega$, $q \le_{n+1} p \longrightarrow q \le_n p$;
(2) For each sequence $(p_n)_{n<\omega}$ in $\mathbb{P}$ satisfying $p_{n+1} \le_n p_n$ for all $n < \omega$, there is some $q \in \mathbb{P}$ such that $q \le_n p_n$ for all $n < \omega$;
(3) For each $p \in \mathbb{P}$, for each pairwise incompatible set $A \subseteq \mathbb{P}$, for each $n < \omega$ there is some $q \le_n p$ such that q is compatible with at most countably many elements of A.

The following property P_f generalizes the Axiom A version of the Laver property, called L_f in [24]. Bartoszyński and Judah proved that the property P_f implies that no Cohen reals are added, and moreover, that the countable support iteration of partial orderings satisfying P_f does not add Cohen reals. (See [24].)

Definition 23 (*[24]*). Let $(\mathbb{P}, \le)$ be a partial ordering satisfying Axiom A and let $f : \omega \to \omega$. $\mathbb{P}$ *satisfies Property* P_f if $\forall p \in \mathbb{P}$, $\forall n, k < \omega$, $\forall A \in [\omega]^{<\omega}$, if $p \Vdash (\dot{B} \subseteq A$ and $|\dot{B}| \le k)$, then $\exists C \subseteq A$ such that $|C| \le k \cdot f(n)$ and $\forall c \notin C$, $\exists q \le_n p$ such that $q \Vdash c \notin \dot{B}$.

The following fact can be proved by an argument analogous to an argument given by Prikry [21] giving a Boolean algebraic equivalent of the property L_f. Recall from that for a given separative partial ordering $\mathbb{P}$, we let $e : \mathbb{P} \to$ r.o.$(\mathbb{P})$ denote the canonical embedding of $\mathbb{P}$ as a dense subset into its Boolean completion.

Fact 24. *Let $\mathbb{P}$ be a separative partial ordering satisfying Axiom A, and let $f : \omega \to \omega$. Then Property P_f holds in $\mathbb{P} \Longleftrightarrow \forall p \in \mathbb{P}$, $\forall n, k < \omega$, $\forall A \in [\omega]^{<\omega}$, if $\dot{B}$ is an* r.o.$(\mathbb{P})$*-name of the form $\langle\{\dot{d}_1, \ldots, \dot{d}_k\}, e(p)\rangle$, where the $\dot{d}_i$ are* r.o.$(\mathbb{P})$*-names for integers, and $e(p) \le \|\dot{B} \subseteq A\|$, then $\exists C \subseteq A$ such that $|C| \le k \cdot f(n)$ and $\forall c \notin C$, $\exists q \le_n p$ such that $e(q) \le \bigwedge_{1 \le i \le k} \|c \ne \dot{d}_i\|$.*

Note: if P_f holds, then f must actually have its range in $\omega \setminus \{0\}$.

The following proof is similar to one given by Prikry [21] that L_f implies the hyper-weak (ω, ω)-distributive law

Proposition 25. *For each separative partial ordering* $(\mathbb{P}, \leq)$ *which satisfies* P_f *for some* $f : \omega \to \omega \setminus \{0\}$*, player II has a winning strategy for* $\mathcal{G}^{\omega}_{\omega-1}$ *in* r.o.$(\mathbb{P})$.

Proof. Suppose P_f holds in $\mathbb{P}$ for some function $f : \omega \to \omega \setminus \{0\}$. Suppose I fixes $a \in \text{r.o.}(\mathbb{P})^+$. Let $p_0 \in \mathbb{P}$ be such that $e(p_0) \leq a$. We show how II can choose at each round of the game to ensure a win.

Suppose we have I and II's choices up to stage n: For each round $i < n$, I has played $\{b_{ij} : j < \omega\}$, a quasi-partition of a, and we have chosen $p_{i+1} \in \mathbb{P}$ and $g(i) \in \omega$ such that $e(p_{i+1}) \leq \bigvee_{j \in \omega \setminus \{g(i)\}} b_{ij}$ and $p_0 \geq_0 p_1 \geq_1 \cdots \geq_{n-1} p_n$.

Round n: Suppose I plays $\{b_{nj} : j < \omega\}$, a quasi-partition of a. Let $A_n = \{0, \ldots, f(n)\}$. For $j < f(n)$, let $c_{nj} = e(p_n) \wedge b_{nj}$, and let $c_{n,f(n)} = e(p_n) \wedge (\bigvee_{f(n) \leq j < \omega} b_{nj})$. Let $\dot{a}_n = \{\langle \check{j}, c_{nj} \rangle : j \leq f(n)\}$ and $\dot{B}_n = \langle \{\dot{a}_n\}, e(p_n) \rangle$. ($k = 1$ here.) Then $\bigvee_{j \leq f(n)} c_{nj} = e(p_n) \leq \| \dot{B}_n \subseteq A_n \|$, so by P_f and Fact 24, $\exists C_n \subseteq A_n$ such that $|C_n| \leq 1 \cdot f(n)$ and $\forall c \notin C_n, \exists q \leq_n p_n$ such that $e(q) \leq \| c \neq \dot{a}_n \|$. Let $g(n)$ be the least element of $A_n \setminus C_n$ and choose a $p_{n+1} \leq_n p_n$ such that $e(p_{n+1}) \leq \| g(n) \neq \dot{a}_n \|$. Then

$$e(p_{n+1}) \leq \| g(n) \neq \dot{a}_n \| \leq \bigvee_{j \in \omega \setminus \{g(n)\}} b_{nj} = a \setminus b_{n,g(n)}. \tag{8}$$

Let II choose to leave out $b_{n,g(n)}$.

In this manner, we obtain a sequence $(p_n)_{n<\omega}$ and a function $g : \omega \to \omega$ such that $\forall n < \omega$, $p_{n+1} \leq_n p_n$ and $e(p_{n+1}) \leq a \setminus b_{n,g(n)}$. By (2) of Axiom A, $\exists q \in \mathbb{P}$ such that $\forall n < \omega, q \leq_n p_n$. By (1) of Axiom A, $\forall n < \omega, q \leq p_n$. Therefore, $\mathbf{0} < e(q) \leq \bigwedge_{n<\omega} a \setminus b_{n,g(n)}$, by (8). Hence, II has a winning strategy for $\mathcal{G}^{\omega}_{\omega-1}$ in r.o.$(\mathbb{P})$. □

Remark 26. In the above proof, property (3) of Axiom A was never used.

Examples 27. The following partial orderings satisfy Axiom A and P_f for some function $f : \omega \to \omega \setminus \{0\}$: Laver, Mathias, Miller, and Random real forcings. (See [24].) Hence, by Proposition 25, II has a winning strategy for $\mathcal{G}^{\omega}_{\omega-1}$ in the Boolean completions of these forcings.

Next, we relate the hyper-weak (κ, λ)-distributive law to the existence of a winning strategy for each of the two players.

Theorem 28. *For each* $\max(\kappa^+, \lambda)$*-complete Boolean algebra, if the hyper-weak* (κ, λ)*-distributive law fails, then I has a winning strategy for* $\mathcal{G}^{\kappa}_{\lambda-1}$.

Proof. Suppose the hyper-weak (κ, λ)-distributive law fails in $\mathbb{B}$. Then by Remark 9 there is a $b > \mathbf{0}$ and a family $\{b_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, of quasi-partitions of b such that $\bigvee_{f:\kappa \to \lambda} \bigwedge_{\alpha<\kappa} \bigvee_{\beta \in \lambda \setminus \{f(\alpha)\}} b_{\alpha\beta} = \mathbf{0}$. I's winning strategy is to play $\{b_{\alpha\beta} : \beta < \lambda\}$ on round α. □

It is not known whether the full converse of Theorem 28 holds in ZFC. Indeed, we conjecture that it does not. However, we do have the following partial converse.

Theorem 29. *For* $\mathbb{B}$ *a* κ^+*-complete Boolean algebra, if I has a winning strategy in* $\mathcal{G}^{\kappa}_{\lambda-1}$*, then the hyper-weak* $(\lambda^{<\kappa}, \lambda)$*-distributive law fails.*

Proof. Suppose σ is a winning strategy for I. Let a be the non-zero element which I fixes according to σ, and let $W_{\langle\,\rangle} = \sigma(\langle\,\rangle)$, the quasi-partition of a of size λ which I plays on round 0 according to σ. Index the elements of $W_{\langle\,\rangle}$ as $\{b_{\langle s(0)\rangle} : s(0) < \lambda\}$. For each $s(0) < \lambda$, let $W_{\langle s(0)\rangle} = \sigma(\langle b_{\langle s(0)\rangle}\rangle)$, the quasi-partition of a which I chooses according to σ if II has just chosen to leave out $b_{\langle s(0)\rangle}$. In general, given $\alpha < \kappa$, $s \in (\lambda)^{\alpha}$, and W_s a quasi-partition of a of size λ, index the elements of W_s as $\{b_{s^\frown s(\alpha)} : s(\alpha) < \lambda\}$. For each $s(\alpha) < \lambda$, let

$$W_{s^\frown s(\alpha)} = \sigma(\langle b_{s \restriction (\beta+1)} : \beta \leq \alpha \rangle), \tag{9}$$

the quasi-partition of a which I chooses according to σ if II has just chosen to leave out $b_{s^\frown s(\alpha)}$. For limit ordinals $\mu < \kappa$ and $s \in (\lambda)^{\mu}$, let

$$W_s = \sigma(\langle b_{s \restriction (\alpha+1)} : \alpha < \mu \rangle). \tag{10}$$

Note that $\{W_s : s \in (\lambda)^{<\kappa}\}$ lists all the possible choices for I under σ.

Claim 30. *The hyper-weak $(\lambda^{<\kappa}, \lambda)$-distributive law fails for W_s, $s \in (\lambda)^{<\kappa}$.*

Let $f : (\lambda)^{<\kappa} \to \lambda$ be given. Recursively define a sequence $t \in (\lambda)^{\kappa}$ by $t \restriction (\alpha + 1) = t \restriction \alpha^\frown f(t \restriction \alpha)$ for each $\alpha < \kappa$. Then $\langle W_{t\restriction\alpha}, b_{t\restriction(\alpha+1)} : \alpha < \kappa\rangle$ is a play of $\mathcal{G}^{\kappa}_{\lambda-1}$ in which I follows the winning strategy σ. Thus,

$$\bigwedge_{s\in(\lambda)^{<\kappa}} a \setminus b_{s^\frown f(s)} \ \leq \bigwedge_{\alpha<\kappa} a \setminus b_{t\restriction(\alpha+1)} = \mathbf{0}. \tag{11}$$

Since f was arbitrary,

$$\bigvee_{f:(\lambda)^{<\kappa}\to\lambda} \ \bigwedge_{s\in(\lambda)^{<\kappa}} a \setminus b_{s^\frown f(s)} = \mathbf{0} \ < \ a \ = \bigwedge_{s\in(\lambda)^{<\kappa}} \ \bigvee_{j<\lambda} b_{s^\frown j}. \quad \square \tag{12}$$

For some pairs of cardinals, Theorems 28 and 29 combine to yield a game-theoretic characterization of the hyper-weak (κ, λ)-distributive law.

Corollary 31. *If $\mathbb{B}$ is κ^+-complete and $\lambda^{<\kappa} = \kappa$, then the hyper-weak (κ, λ)-distributive law holds in $\mathbb{B}$ iff I does not have a winning strategy for $\mathcal{G}^{\kappa}_{\lambda-1}$ played in $\mathbb{B}$.*

In particular, this yields a game-theoretic characterization of the hyper-weak (ω, ω)-distributive law for Boolean σ-algebras.

Corollary 32 (*GCH*). *Suppose $\mathbb{B}$ is κ^+-complete and either* (a) $\lambda < \kappa$, *or* (b) $\lambda = \kappa$ *and κ is regular. Then the hyper-weak (κ, λ)-distributive law holds in $\mathbb{B}$ iff I does not have a winning strategy for $\mathcal{G}^{\kappa}_{\lambda-1}$ played in $\mathbb{B}$.*

4. Constructions of κ^+-Suslin algebras in which many games are undetermined

In this section, we show that it is consistent with ZFC that every game of the sort considered in this paper can be undetermined (see Corollary 36). For each infinite cardinal κ and each infinite regular cardinal $\nu \leq \mathrm{cf}(\kappa)$, we will construct a κ^+-Suslin tree T such that for all ρ with $\nu \leq \rho \leq \kappa$, all games of length ρ are undetermined. We start by recalling some basic notation. For $t \in T$, $\mathrm{ht}(t)$ denotes o.t.$(\{s \in T : s <_T t\})$. For $\alpha < \kappa^+$, let $\mathrm{Lev}(\alpha) = \{t \in T : \mathrm{ht}(t) = \alpha\}$, the *$\alpha$-th level of T*, and $T_\alpha = \{t \in T : \mathrm{ht}(t) < \alpha\}$. Let T^* denote T under the reverse partial ordering $\geq_T$, and let $e : T^* \to \mathrm{r.o.}(T^*)$ denote the canonical embedding of T^* into its Boolean completion. We will construct T so that each strategy for II for the game $\mathcal{G}^{\nu}_{\kappa-1}$ in $\mathrm{r.o.}(T^*)$ will fail to be winning when I plays some particular sequence of partitions of unity in $\mathrm{r.o.}(T^*)$.

Theorem 33. *Let κ be any infinite cardinal, and let ν be any regular cardinal such that $\omega \leq \nu \leq \mathrm{cf}(\kappa)$ and $\kappa^{<\nu} = \kappa$. Suppose that $\square_\kappa$ holds, and for every stationary $S \subseteq \kappa^+ \cap \mathrm{cof}(\nu)$, $\diamondsuit_{\kappa^+}(S)$ holds. Then there is a κ^+-Suslin algebra which contains a $< \nu$-closed dense subset and in which II does not have a winning strategy for $\mathcal{G}^{\nu}_{\kappa-1}$.*

Proof. Let $\mathrm{cof}(\nu)$ denote $\{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \nu\}$. For any set of ordinals X, let $\lim(X)$ denote the set of ordinals $\alpha \in X$ such that α is the limit of an infinite, strictly increasing sequence of elements of X. Using a $\square_\kappa$-sequence, we construct in the usual manner (see for example Devlin's book on L [25]) another $\square_\kappa$-sequence $\langle D_\alpha : \alpha \in \lim(\kappa^+)\rangle$ and a non-reflecting stationary set $S \subseteq \mathrm{cof}(\nu)$ such that for all $\alpha \in \lim(\kappa^+)$

(1) $D_\alpha \subseteq \alpha$ is club in α;
(2) $\mathrm{cf}(\alpha) < \kappa \longrightarrow \mathrm{o.t.}(D_\alpha) < \kappa$;
(3) $\gamma \in \lim(D_\alpha) \longrightarrow D_\gamma = D_\alpha \cap \gamma$;
(4) $\lim(D_\alpha) \cap S = \emptyset$.

We fix some definitions and notation. Fix a $\diamondsuit_{\kappa^+}(S)$-sequence $\langle A_\alpha : \alpha \in S\rangle$; that is, a sequence such that $A_\alpha \subseteq \alpha$, and $\{\alpha \in S : A_\alpha = A \cap \alpha\}$ is stationary for all $A \subseteq \kappa^+$. We say that $\langle P(\alpha, \gamma) : \gamma < \kappa\rangle$ is a *partition of* $\mathrm{Lev}(\alpha)$ *into κ-many pieces* if $\forall \gamma < \gamma' < \kappa$, $P(\alpha, \gamma) \subseteq \mathrm{Lev}(\alpha)$, $P(\alpha, \gamma) \neq \emptyset$, $P(\alpha, \gamma) \cap P(\alpha, \gamma') = \emptyset$, and $\bigcup_{\gamma<\kappa} P(\alpha, \gamma) = \mathrm{Lev}(\alpha)$.

As we build the tree, for each successor $\alpha < \kappa^+$ we will construct a partition $\mathcal{P}_\alpha = \langle P(\alpha, \gamma) : \gamma < \kappa\rangle$ of $\mathrm{Lev}(\alpha)$ into κ-many pieces. To each such partition $\mathcal{P}_\alpha$ we may associate a partition of unity $\{\bigvee\{e(t) : t \in P(\alpha, \gamma)\} : \gamma < \kappa\}$

in r.o.(T^*). In the discussion that follows we abuse notation and say "I plays $\mathcal{P}_\alpha$" when formally we mean that I plays the corresponding partition of unity in r.o.(T^*).

After we have built the tree T, we will show that player II has no winning strategy in for $\mathcal{G}^\nu_{\kappa-1}$ in the following way. Given a strategy for player II we will show how player I can play some sequence $\langle \mathcal{P}_{\alpha_i} : i < \nu\rangle$ for $\langle \alpha_i : i < \nu\rangle$ a strictly increasing sequence of successor ordinals in κ^+, and win against II's strategy. Fix a bijection $\varphi : (\kappa^+)^{<\nu} \times \kappa \to \kappa^+$. φ will code all the possible partial strategies for II when I is restricted to playing partitions from among the collection $\{\mathcal{P}_{\alpha+1} : \alpha < \kappa^+\}$ as subsets of κ^+.

Construction of $(T, \leq_T)$ and $\{\mathcal{P}_{\alpha+1} : \alpha < \kappa^+\}$.

We will construct the tree T to be *normal*: that is to say

(1) T has a unique point on level 0.
(2) T has *unique limits*, that is for every limit α and every $t \in \mathrm{Lev}(\alpha)$, t is determined by $\{s \in T : s <_T t\}$.
(3) For all α, β with $\alpha < \beta < \kappa^+$, for every $s \in \mathrm{Lev}(\alpha)$ there is at least one $t \in \mathrm{Lev}(\beta)$ with $s <_T t$.

Let $\mathrm{Lev}(0) = \{0\}$. Let $\mathrm{Lev}(1) = (\kappa \cdot 2) \setminus \{0\}$, and let $\mathcal{P}_1 = \langle P(1, \gamma) : \gamma < \kappa\rangle$ be some partition of Lev(1) into κ-many pieces. Suppose $\alpha \geq 1$ and $\mathrm{Lev}(\alpha)$ has been constructed. Put κ-many immediate successors above each element of $\mathrm{Lev}(\alpha)$ in such a way that $\mathrm{Lev}(\alpha+1) = \{\beta < \kappa^+ : \kappa\cdot(\alpha+1) \leq \beta < \kappa\cdot(\alpha+2)\}$. Let $\mathcal{P}_{\alpha+1} = \langle P(\alpha+1, \gamma) : \gamma < \kappa\rangle$ be a partition of $\mathrm{Lev}(\alpha+1)$ into κ-many pieces such that for each node t in $\mathrm{Lev}(\alpha)$ and each $\gamma < \kappa$, $\mathrm{Lev}(\alpha+1) \cap P(\alpha+1, \gamma)$ contains exactly one immediate successor of t.

Now suppose $\alpha < \kappa^+$ is a limit ordinal and T_α has been constructed. Important to the construction of $\mathrm{Lev}(\alpha)$ is the following notion:

Before building level α we will associate to each $t \in T_\alpha$ the *canonical α-branch for t in T_α*, this is a branch of T_α passing through t which we denote by $B^{\alpha,t}$. To do this we will define for each $\gamma \in D_\alpha \setminus (ht(t) + 1)$ an element $\beta^{\alpha,t}_\gamma$ on level γ so that

$$\{t\} \cup \{\beta^{\alpha,t}_\gamma : \gamma \in D_\alpha \setminus (ht(t) + 1)\}$$

is linearly ordered in T.

The construction is simple: we just choose $\beta^{\alpha,t}_\gamma$ to be the least (in the ordering of the ordinals) element on level γ which is above all elements of

$$t \cup \{\beta^{\alpha,t}_\zeta : \zeta \in (D_\alpha \cap \gamma) \setminus (ht(t) + 1)\}.$$

There is no problem when γ is successor in D_α because the tree is normal. The construction will be able to proceed at limits because at every limit $\eta \notin S$ we will make sure that a point on level η is put over $B^{t,\eta}$; now if γ is limit in D_α we know that $D_\gamma = D_\alpha \cap \gamma$, the uniform construction of the canonical branches ensures that $B^{\gamma,t}$ is an initial segment of $B^{\alpha,t}$, so the point which was over $B^{\gamma,t}$ at stage γ gives us the unique choice for $\beta^{\alpha,t}_\gamma$. The key point is that limit points of D_α are not in S, and it is only at stages in S where we take steps to kill antichains and are obliged not to complete certain branches.

Case 1. $\mathrm{cf}(\alpha) < \nu$. Then extend every α-branch of T_α to level α with exactly one extension in such a way that $\mathrm{Lev}(\alpha) = \{\beta < \kappa^+ : \kappa \cdot \alpha \leq \beta < \kappa \cdot (\alpha + 1)\}$. In particular, for each $t \in T_\alpha$, the canonical branch B^α_t is extended to $\mathrm{Lev}(\alpha)$. By our cardinal arithmetic assumption $\kappa^{cf(\alpha)} = \kappa$, so there are only at most κ branches and we are not obliged to create too many points on level α.

Case 2. $\mathrm{cf}(\alpha) \geq \nu$ and $\alpha \notin S$. For each $t \in T_\alpha$, extend the canonical α-branch B^α_t to $\mathrm{Lev}(\alpha)$ with exactly one extension so that $\mathrm{Lev}(\alpha) = \{\beta < \kappa^+ : \kappa \cdot \alpha \leq \beta < \kappa \cdot (\alpha + 1)\}$.

Case 3. $\alpha \in S$. Let (C) be the statement, "$T_\alpha = \alpha$ and $\varphi''((\alpha)^{<\nu} \times \kappa) = \alpha$." Let (M) be the statement, "A_α is a maximal antichain in T_α." Let (F) be the statement, "$\varphi^{-1}(A_\alpha)$ is a function, and $\mathrm{dom}(\varphi^{-1}(A_\alpha)) = (\alpha)^{<\nu}$." If either (C) fails, or both (M) and (F) fail, then extend all the canonical α-branches of T_α to $\mathrm{Lev}(\alpha)$. If (C) and (M) hold and (F) fails, then for each $t \in T_\alpha$, choose one r in T_α such that $r \geq_T t$ and $r \geq_T u$ for some $u \in A_\alpha$. Extend the canonical α-branch B^α_r to $\mathrm{Lev}(\alpha)$.

Now suppose (C) and (F) both hold. Let $f = \varphi^{-1}(A_\alpha)$. Then $f : (\alpha)^{<\nu} \to \kappa$ can be interpreted as a partial strategy which tells II what to do whenever I plays a sequence of partitions from among $\{\mathcal{P}_{\beta+1} : \beta < \alpha\}$. Fix a strictly increasing sequence $\langle \alpha_i : i < \nu \rangle$ such that each α_i is a successor ordinal and $\sup_{i<\nu} \alpha_i = \alpha$. This is possible since $\alpha \in S$ implies $\mathrm{cf}(\alpha) = \nu$. We let I play $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$ and II play according to f. For each $i < \nu$, let

$$\gamma_i = f(\langle \alpha_j : j \leq i \rangle). \tag{13}$$

Then when I plays the sequence $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$ and II plays by f, II chooses to leave out $P(\alpha_i, \gamma_i)$ on round i.

Let $t \in T_\alpha$. If (M) holds, let $r \in T_\alpha$ be such that $r \geq_T t$ and $r \geq_T u$ for some $u \in A_\alpha$. Otherwise, let $r = t$. Let $i(t) < \nu$ be the least ordinal such that $\alpha_{i(t)} > \mathrm{ht}(r)$. We will construct a chain $\langle x_i : i(t) \leq i < \nu \rangle$ in T_α such that $x_{i(t)} \geq_T r$; for each $i(t) \leq i < j < \nu$, $x_i <_T x_j$; and for each $i(t) \leq i < \nu$, $x_i \in P(\alpha_i, \gamma_i)$. Let $p_{i(t)}$ be the least ordinal in $\mathrm{Lev}(\alpha_{i(t)} - 1)$ such that $p_{i(t)} >_T r$. Let $x_{i(t)}$ be the immediate successor of $p_{i(t)}$ in $P(\alpha_{i(t)}, \gamma_{i(t)})$. For $i(t) < i = j + 1 < \nu$, let p_i be the least ordinal in $\mathrm{Lev}(\alpha_i - 1)$ such that $p_i \geq_T x_j$. Let x_i be the immediate successor of p_i in $P(\alpha_i, \gamma_i)$. For i a limit ordinal with $i(t) < i < \nu$, let $\lambda_i = \sup_{j<i} \alpha_j$. Then λ_i is a limit ordinal with $\mathrm{cf}(\lambda_i) < \nu$, so every branch in T_{λ_i} has an extension in $\mathrm{Lev}(\lambda_i)$, by Case 1. Let p'_i be the least ordinal in $\mathrm{Lev}(\lambda_i)$ such that $p'_i > x_j$ for all $i(t) \leq j < i$. α_i is a successor ordinal greater than λ_i, so let p_i be the least ordinal in $\mathrm{Lev}(\alpha_i - 1)$ such that $p_i \geq_T p'_i$. Let x_i be the immediate successor of p_i in $P(\alpha_i, \gamma_i)$. Let $b^\alpha_t = \{s \in T_\alpha :$ for some i with $i(t) \leq i < \nu,\ s <_T x_i\}$. b^α_t is an α-branch in T_α which passes through the pieces of the partitions which II chooses to leave out on each level α_i according to f. For each $t \in T_\alpha$, extend b^α_t to $\mathrm{Lev}(\alpha)$.

For each of the subcases of Case 3, for each $t \in T_\alpha$ we have chosen one α-branch containing t to extend to $\mathrm{Lev}(\alpha)$. Extend each of these branches uniquely to $\mathrm{Lev}(\alpha)$ in such a way that $\mathrm{Lev}(\alpha) = \{\beta < \kappa^+ : \kappa \cdot \alpha \leq \beta < \kappa \cdot (\alpha + 1)\}$.

Let $T = \bigcup_{\alpha<\kappa} \mathrm{Lev}(\alpha)$. This concludes the construction of $(T, \leq_T)$. By the usual argument, $(T, \leq_T)$ is a κ^+-Suslin tree. Let $C_T = \{\alpha < \kappa^+ : T_\alpha = \alpha\}$ and $C_\varphi = \{\alpha < \kappa^+ : \varphi''((\alpha)^{<\nu} \times \kappa) = \alpha\}$. C_T is a club subset of κ^+. C_φ is unbounded in κ^+ and closed under increasing sequences of length ν.

Claim 34. *II does not have a winning strategy for $\mathcal{G}^\nu_{\kappa-1}$ played on* r.o.(T^*).

Let $h : (\kappa^+)^{<\nu} \to \kappa$ represent a strategy for II in $\mathcal{G}^\nu_{\kappa-1}$ when I plays partitions from among $\{\mathcal{P}_{\alpha+1} : \alpha < \kappa^+\}$. $\{\beta \in S : A_\beta = \varphi''(h) \cap \beta\}$ is a stationary subset of $\mathrm{cof}(\nu)$. Hence, there exists an $\alpha \in \{\beta \in S : A_\beta = \varphi''(h) \cap \beta\} \cap C_\varphi \cap C_T$. Let $f = \varphi^{-1}(A_\alpha)$. Then $f = h \restriction (\alpha)^{<\nu}$. $\alpha \in \{\beta \in S : A_\beta = \varphi''(h) \cap \beta\} \cap C_\varphi \cap C_T$ implies $\mathrm{Lev}(\alpha)$ was constructed according to Case 3 with statements (C) and (F) holding for f. Let $\langle \alpha_i : i < \nu \rangle$ be the strictly increasing sequence we picked such that each α_i is a successor ordinal and $\sup_{i<\nu} \alpha_i = \alpha$. Let I play the sequence $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$. Let $q \in \mathrm{Lev}(\alpha)$. Then there is some $t \in T_\alpha$ such that q is the unique extension of the α-branch b^α_t. By our construction, the unique element $x_{i(t)} \in \mathrm{Lev}(\alpha_{i(t)}) \cap b^\alpha_t$ is in $P(\alpha_{i(t)}, \gamma_{i(t)})$, the piece of the partition $\mathcal{P}_{\alpha_{i(t)}}$ which f chooses to leave out on round $i(t)$ when I has played the sequence $\langle \mathcal{P}_{\alpha_j} : j \leq i(t) \rangle$. Since q was an arbitrary member of $\mathrm{Lev}(\alpha)$, there is no $(\alpha + 1)$-branch in T which misses all of the pieces of the partitions $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$ which f chose to leave out. Hence, if II plays according to the strategy h, II loses when I plays the sequence $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$. Thus, h is not a winning strategy for II. Since h was arbitrary, II does not have a winning strategy for $\mathcal{G}^\nu_{\kappa-1}$ in r.o.(T^*). $\square$

Remark 35. (1) In Case 3 when (C) and (F) hold for α, for each $t \in T_\alpha$ we only needed to extend some α-branch containing the element $x_i \in P(\alpha_{i(t)}, \gamma_{i(t)})$ to ensure that II does not have a winning strategy for $\mathcal{G}^\nu_{\kappa-1}$ in r.o.(T^*). Our construction shows that II does not have a winning strategy in r.o.(T^*) in the following game, which is even weaker than $\mathcal{G}^\nu_{\kappa-1}$: Players I and II play the game $\mathcal{G}^\nu_{\kappa-1}$ constructing a sequence $\langle a, W_\alpha, b_\alpha : \alpha < \nu \rangle$. After the play is over, II gets to choose some set $X \subseteq \nu$ of cardinality ν. I wins the play iff $\bigwedge_{\alpha \in X} \bigvee (W_\alpha \setminus \{b_\alpha\}) = \mathbf{0}$.

(2) In the case when $\kappa^{<\kappa} = \kappa$, the previous construction can be slightly modified so that using only $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$ we can construct a κ^+-Suslin algebra $\mathbb{B}$ which contains a $< \kappa$-closed dense subset, and in which the game $\mathcal{G}^\kappa_{\lambda-1}$ is undetermined for each $\omega \leq \lambda \leq \kappa$. Hence, also for each λ, μ with $2 \leq \mu \leq \min(\kappa, \lambda)$, the game $\mathcal{G}^\kappa_{<\mu}(\lambda)$ is undetermined in $\mathbb{B}$.

(3) The referee has pointed out that the principle $\square^*_\kappa$ could be used in place of $\square_\kappa$.

Corollary 36. *Let κ be any infinite cardinal, and let ν be any regular cardinal such that $\omega \leq \nu \leq \mathrm{cf}(\kappa)$ and $\kappa^{<\nu} = \kappa$. Suppose that $\square_\kappa$ holds and $\diamondsuit_{\kappa^+}(S)$ holds for every stationary set $S \subseteq \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \nu\}$. Then there is a κ^+-Suslin algebra which contains a $< \nu$-closed dense subset, and in which for each ρ, λ with $\nu \leq \rho \leq \mathrm{cf}(\kappa)$ and $\omega \leq \lambda \leq \kappa$ the game $\mathcal{G}^\rho_{\lambda-1}$ is undetermined. Hence, for every ρ, λ, μ with $\nu \leq \rho \leq \mathrm{cf}(\kappa)$ and $2 \leq \mu \leq \min(\kappa, \lambda)$, the game $\mathcal{G}^\rho_{<\mu}(\lambda)$ is undetermined.*

Proof. Let $\mathbb{B}$ denote r.o.(T^*), where T is the κ^+-Suslin tree constructed in Theorem 33. Since the (κ, ∞)-distributive law holds in every κ^+-Suslin algebra, I does not have a winning strategy for $\mathcal{G}_1^\kappa(\infty)$ in $\mathbb{B}$, by Corollary 1.6 of [17]. Hence, by Fact 20, for each infinite cardinal $\rho \le \kappa$, for every λ, μ with $2 \le \mu \le \lambda$, I does not have a winning strategy for $\mathcal{G}_{<\mu}^\rho(\lambda)$ in $\mathbb{B}$; and for every $\lambda \ge \omega$, I does not have a winning strategy for $\mathcal{G}_{\lambda-1}^\rho$ in $\mathbb{B}$. Since II does not have a winning strategy for $\mathcal{G}_{\kappa-1}^\nu$, Fact 20 implies that for each $\nu \le \rho \le \kappa$, for every $\omega \le \lambda \le \kappa$, II does not have a winning strategy for $\mathcal{G}_{\lambda-1}^\rho$ in $\mathbb{B}$. Hence, for every λ, μ with $2 \le \mu \le \min(\lambda, \kappa)$, II also does not have a winning strategy for $\mathcal{G}_{<\mu}^\rho(\lambda)$ in $\mathbb{B}$. Moreover, since $\mathbb{B}$ contains a dense $<\nu$-closed subset, II wins $\mathcal{G}_1^\theta(\infty)$ for all $\theta < \nu$, and II even wins Foreman's game $G_{\theta^+}^{\mathrm{I}}$ for all $\theta < \nu$ (see [23]). □

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