Topological Ramsey spaces in creature forcing

Natasha Dobrinen

University of Denver

CMO, 2016

Motivation

Observation (Todorcevic). There are strong connections between creature forcing and topological Ramsey spaces deserving of a systematic study.

Question. Which creature forcings are essentially topological Ramsey spaces?

Topological Ramsey spaces dense in forcings make possible exact results for Rudin-Keisler and Tukey structures on ultrafilters forced by

1. $\mathbb{P}_\alpha$, $\alpha < \omega_1$, of Laflamme in [D/Todorcevic 2014, 2015 TAMS];
2. Forcings of Baumgartner and Taylor, of Blass, and others in [D/Mijares/Trujillo AFML];
3. $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$, $2 \le \alpha < \omega_1$ in [D 2015 JSL, 2016 JML].

Moreover, the forced ultrafilters have *complete combinatorics* over $L(\mathbb{R})$ in the presence of a supercompact cardinal [Di Prisco/Mijares/Nieto].

Some Results of Rosłanowski and Shelah

Partition theorems from creatures and idempotent ultrafilters, [Rosłanowski/Shelah 2013] seemed a good place to start this investigation.

$\mathbf{H}$ denotes any function with $\mathrm{dom}(\mathbf{H}) = \omega$ such that $\mathbf{H}(n)$ is a finite non-empty set for each $n < \omega$.

$$\mathcal{F}_{\mathbf{H}} = \bigcup_{u \in \mathrm{FIN}} \prod_{n \in u} \mathbf{H}(n).$$

pure candidates are certain infinite sequences $\bar{t}$ of creatures (defined later in context). $\mathrm{pos}(\bar{t})$ is a subset of $\mathcal{F}_{\mathbf{H}}$ determined by $\bar{t}$.

Thm. [R/S] Under certain hypotheses on a creature forcing, given a pure candidate $\bar{t}$ and a coloring $c : \mathrm{pos}(\bar{t}) \to 2$ there is a pure candidate $\bar{s}$ stronger than $\bar{t}$ such that c is constant on $\mathrm{pos}(\bar{s})$.

Cor. [R/S] (CH) There is an ultrafilter $\mathcal{U}$ on base set $\mathcal{F}_{\mathsf{H}}$ generated by $\{\mathrm{pos}(\bar{t}_\alpha) : \alpha < \omega_1\}$ for a decreasing sequence of pure candidates $\langle \bar{t}_\alpha : \alpha < \omega_1 \rangle$, moreover, satisfying the previous partition theorem: For any $\bar{t}$ such that $\mathrm{pos}(\bar{t}) \in \mathcal{U}$ and any partition of $\mathrm{pos}(\bar{t})$ into finitely many pieces, there is a pure candidate $\bar{s} \le \bar{t}$ such that $\mathrm{pos}(\bar{s})$ is contained in one piece of the partition and $\mathrm{pos}(\bar{s}) \in \mathcal{U}$.

Remark. This is similar to the construction of a stable-ordered union ultrafilter on FIN using Hindman's Theorem.

Remark. The proofs in [R/S] use the Galvin-Glazer method extended to certain classes of creature forcings.

Thm. [D] Three examples of sets of pure candidates in [R/S] contain dense subsets forming topological Ramsey spaces.

For two of these spaces, the pigeonhole prinicples rely on the following product tree Ramsey theorem.

New Product Tree Ramsey Theorem

For $p \le n$, $[K_p]^k \times \prod_{j \in (n+1) \setminus \{p\}} K_j$ denotes

$$K_0 \times \cdots \times K_{p-1} \times [K_p]^k \times K_{p+1} \times \cdots \times K_n.$$

Thm. [D] Given $k \ge 1$, a sequence of positive integers $(m_0, m_1, \dots)$, sets K_j, $j < \omega$ such that $|K_j| \ge j+1$, and a coloring

$$c : \bigcup_{n<\omega} \bigcup_{p \le n} ([K_p]^k \times \prod_{j \in (n+1) \setminus \{p\}} K_j) \to 2,$$

there are infinite sets $L, N \subseteq \omega$ such that, enumerating L and N in increasing order, $l_0 \le n_0 < l_1 \le n_1 < \dots$, and there are subsets $H_j \subseteq K_j$, $j < \omega$, such that $|H_{l_i}| = m_i$ for each $i < \omega$, $|H_j| = 1$ for each $j \in \omega \setminus L$, and c is constant on

$$\bigcup_{n \in N} \bigcup_{l \in L \cap (n+1)} ([H_l]^k \times \prod_{j \in (n+1) \setminus \{l\}} H_j).$$

The proof of this theorem uses

1. proof methods of a theorem of Di Prisco, Llopis, and Todorcevic to produce an augmented version of their theorem,
2. prudent diagonalizations,
3. and a final application of the theorem of Di Prisco, Llopis, and Todorcevic to the stems after the diagonalizations.

Thm. [DiPrisco/Llopis/Todorcevic 2004] There is an $R : (\mathbb{N}^+)^{<\omega} \to \mathbb{N}^+$ such that for every infinite sequence $(m_j)_{j<\omega}$ of positive integers and for every coloring

$$c : \bigcup_{n<\omega} \prod_{j\leq n} R(m_0, \ldots, m_j) \to 2,$$

there exist $H_j \subseteq R(m_0, \ldots, m_j)$, $|H_j| = m_j$, for $j < \omega$, such that c is constant on the product

$$\prod_{j\leq n} H_j$$

for infinitely many $n < \omega$.

Remark. The difference is that we need sets of size k to be able to move up and down indices of the product.

As an intermediate step we prove

Thm. [D] Let $L, N \in [\omega]^\omega$ such that $l_0 \leq n_0 < l_1 \leq n_1 < \dots$.
Let $k, m_0 \geq 1$, and let K_j ($j \geq l_0$) be nonempty sets with $|K_{l_0}| = R_k(m_0)$, $|K_{l_i}| \geq i$ ($i \geq 1$), and $|K_j| = 1$ for each $j \in (l_0, \omega) \setminus L$.
Then for each coloring

$$c : \bigcup_{n \in N} ([K_{l_0}]^k \times \prod_{j \in (l_0, n]} K_j) \to 2,$$

and each $r < \omega$, there are infinite $L' \subseteq L$, $N' \subseteq N$ with $l'_0 = l_0 \leq n'_0 < l'_1 \leq n'_1 < \dots$, and there are $H_j \subseteq K_j$ such that $|H_{l_0}| = m_0$, $|H_{l'_i}| = r + i$ for each $i \geq 1$, $|H_j| = 1$ for each $j \in (l_0, \omega) \setminus L'$, and c is constant on

$$\bigcup_{n \in N'} ([H_{l_0}]^k \times \prod_{j \in (l_0, n]} H_j).$$

Then diagonalize to make the k-size subsets range over all indices, and apply Theorem [DLT] to finite stems obtain our theorem.

We now look at a specific example of a creature forcing.

Example 2.10 in [Roslanowski/Shelah 2013]

$\mathbf{H}_1(n) = n + 1$, for each $n < \omega$.

$\mathcal{F}_{\mathbf{H}_1} = \{\text{functions } f : \text{dom}(f) \text{ is finite and } \forall n \in \text{dom}(f)(f(n) \leq n)\}$.

K_1 = set of all creatures $t = (\mathbf{nor}[t], \mathbf{val}[t], \mathbf{dis}[t], m^t_{\rm dn}, m^t_{\rm up})$ such that

- $\mathbf{dis}[t] = (u^t, i^t, A^t)$, where $u^t \subseteq [m^t_{\rm dn}, m^t_{\rm up})$, $i^t \in u^t$, $\emptyset \neq A^t \subseteq \mathbf{H}_1(i^t) = i^t + 1$,
- $\mathbf{nor}[t] = \log_2(|A^t|)$,
- $\mathbf{val}[t] \subseteq \prod_{j\in u} \mathbf{H}_1(j) = \prod_{j \in u}(j+1)$ s.t. $\{f(i^t) : f \in \mathbf{val}[t]\} = A^t$.

The Sub-Composition Operation: For $t_0, \dots, t_n \in K_1$ with $m^{t_l}_{\mathrm{up}} = m^{t_{l+1}}_{\mathrm{dn}}$ for all $l \le n$,

$\Sigma^*_1(t_0, \dots, t_n)$ is all $t \in K_1$ such that $m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$, $m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}}$, and

$$u^t = \bigcup_{j \le n} u^{t_j}, \quad i^t = i^{t_l}, \quad A^t \subseteq A^{t_l} \text{ for some } l \le n,$$

and $\mathbf{val}[t] \subseteq \{f_0 \cup \dots \cup f_n : (f_0, \dots, f_n) \in \mathbf{val}[t_0] \times \dots \times \mathbf{val}[t_n]\}$.

$\mathrm{PC}^{\mathtt{tt}}_\infty(K_1, \Sigma^*_1)$ denotes the set of all **pure candidates** $\bar{t} = (t_0, t_1, \dots)$ such that for each $n < \omega$, $t_n \in K_1$ and $m^{t_n}_{\mathrm{up}} = m^{t_{n+1}}_{\mathrm{dn}}$, and $\lim_{n \to \infty} \mathbf{nor}[t_n] = \infty$.

$\bar{s} \le \bar{t}$ iff $\exists\ (n_j)_{j<\omega}$ strictly increasing s.t. $\forall j$, $s_j \in \Sigma^*_1(t_{n_j}, \dots, t_{n_{j+1}-1})$.

The set of *possibilities* on the pure candidate $\bar{t}$ is

$$\mathrm{pos}^{\mathtt{tt}}(\bar{t}) = \bigcup\{f_0 \cup \dots \cup f_n : n \in \omega \wedge \forall i \leq n\,(f_i \in \mathbf{val}[t_i])\}. \qquad (1)$$

Thm. [R/S] Given $\bar{t} \in \mathsf{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma^*_1)$, $l \geq 1$, $d_k : \mathrm{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright k) \to l$, $k < \omega$, $\exists \bar{s} \leq \bar{t}$ in $\mathsf{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma^*_1)$ and a $l' < l$ such that for each $i < \omega$, if k is such that $s_i \in \Sigma^*_1(\bar{t} \upharpoonright k)$ and $f \in \mathrm{pos}^{\mathtt{tt}}(\bar{s} \upharpoonright i)$, then $d_k(f) = l'$.

$\bar{t} \upharpoonright n$ denotes $(t_n, t_{n+1}, \dots)$.

Remark. This theorem will be recovered from showing that there is a topological Ramsey space dense in $\mathsf{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma^*_1)$.

A dense subset of $\mathrm{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma_1^*)$ forming a tRs

Recall: $\mathbf{H}_1(n) = n + 1$.

Creatures $t \in K_1$ are determined by $m^t_{\mathrm{dn}} < m^t_{\mathrm{up}}$, $u^t \subseteq [m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})$, $i^t \in u^t$, $A^t \subseteq \mathbf{H}_1(i^t)$, $\mathbf{val}[t] \subseteq \prod_{j \in u^t} \mathbf{H}_1(j)$ satisfying $\{f(i^t) : f \in \mathbf{val}[t]\} = A^t$.

$\mathcal{R}(K_1, \Sigma_1)$ is the set of $\bar{t} = (t_n : n < \omega) \in \mathrm{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma_1^*)$ such that $\forall n$,

1. $|A^{t_n}| = n + 1$ and
2. for each $a \in A^{t_n}$, there is exactly one function $g^{t_n}_a \in \mathbf{val}[t_n]$ such that $g_a(i^{t_n}) = a$.

Thus, $\mathbf{val}[t_n] = \{g^{t_n}_a : a \in A^{t_n}\}$ and $|\mathbf{val}[t_n]| = |A^{t_n}| = n + 1$.

A dense subset of $\mathrm{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma_1^*)$ forming a tRs

For $k < \omega$ and $\bar{s} = (s_0, s_1, \dots) \in \mathcal{R}(K_1, \Sigma_1^*)$, $r_k(\bar{s}) = (s_0, \dots, s_{k-1})$.

Thm. [D] $(\mathcal{R}(K_1, \Sigma_1^*), \leq, r)$ is a topological Ramsey space which is dense in $\mathrm{PC}^{\mathtt{tt}}_{\infty}(K_1, \Sigma_1^*)$.

Cor. [D] Given $\bar{t} \in \mathcal{R}(K_1, \Sigma_1^*)$ and $c_k : \mathcal{AR}_k|\bar{t} \to l$ for each $k \geq 1$, there is an $\bar{s} \leq \bar{t}$ in $\mathcal{R}(K_1, \Sigma_1^*)$ and an $l' < l$ such that for each k, c_k is constantly l' on $r_k[k-1, \bar{s}]$.

Using the fact that for $\bar{t} \in \mathcal{R}(K_1, \Sigma_1^*)$, $|\mathrm{pos}^{\mathtt{tt}}(t_n)| = n + 1$ for each n, we can quickly derive Rosłanowski and Shelah's result for this example, and hence obtain an ultrafilter on $\mathcal{F}_{\mathbf{H}_1}$ which satisfies the partition theorem of [R/S].

The proof that $(\mathcal{R}(K_1, \Sigma_1^*), \leq, r)$ is a topological Ramsey space hinges on proving the pigeonhole principle **Axiom A.4**:

Given $\bar{t} \in \mathcal{R}(K_1, \Sigma_1^*)$, $k \geq 1$, and $c : r_k[k-1, \bar{t}\,] \to 2$, there is an $\bar{s} \leq \bar{t}$ such that c is constant on $r_k[k-1, \bar{s}]$.

Members $(t_0, \ldots, t_{k-2}, x)$ of $r_k[k-1, \bar{t}\,]$ are completely determined by the triple $(i^x, A^x, m^x_{\mathrm{up}})$. So c is really coloring

$$\bigcup_{n \geq k-1} \bigcup_{k-1 \leq p \leq n} A^{t_{k-1}} \times \cdots \times A^{t_{p-1}} \times [A^{t_p}]^k \times A^{t_{p+1}} \times A^{t_n}.$$

Applying our Product Tree Ramsey Theorem yields the pigeonhole principle.

Example 2.11 in [Roslanowski/Shelah 2013]

$\mathbf{H}_2(n) = 2$ for $n < \omega$. $\mathcal{F}_{\mathbf{H}_2} = \{f : \mathrm{dom}(f) \in \mathrm{FIN} \text{ and } f : \mathrm{dom}(f) \to 2\}$.

K_2 = set of all creatures $t = (\mathbf{nor}[t], \mathbf{val}[t], \mathbf{dis}[t], m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})$ such that

- $\emptyset \neq \mathbf{dis}[t] \subseteq [m^t_{\mathrm{dn}}, m^t_{\mathrm{up}})$,
- $\mathbf{val}[t] \subseteq {}^{\mathbf{dis}[t]}2$,
- $\mathbf{nor}[t] = \log_2(|\,\mathbf{val}[t]|)$.

For $t_0, \dots, t_n \in K_2$ with $m^{t_l}_{\mathrm{up}} \leq m^{t_{l+1}}_{\mathrm{dn}}$ for all $l \leq n$, $\Sigma_2(t_0, \dots, t_n)$ consists of all creatures $t \in K_2$ such that

$$m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}},\ m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}},\ \mathbf{dis}[t] = \mathbf{dis}[t_l], \mathbf{val}[t] \subseteq \mathbf{val}[t_l], \text{for some } l \leq n.$$

$\mathsf{PC}_\infty(K_2, \Sigma_2)$ denotes the set of all **pure candidates** $\bar{t} = (t_0, t_1, \dots)$ such that for each $i < \omega$, $t_i \in K_2$ and $m_{\rm up}^{t_i} \le m_{\rm dn}^{t_i}$, and $\lim_{i\to\infty} \mathbf{nor}[t_i] = \infty$.

$\bar{s} \le \bar{t}$ iff $\exists\ (j_n)_{n<\omega}$ strictly increasing s.t. $\forall n$, $s_n \in \Sigma_2(t_{j_{2n}}, \dots, t_{j_{2n+1}})$.

The set of *possibilities* on the pure candidate $\bar{t}$ is

$$\mathrm{pos}(\bar{t}) = \bigcup \{\mathbf{val}[t_n] : n < \omega\}. \qquad (2)$$

Thm. [R/S] Given $\bar{t} \in \mathsf{PC}_\infty(K_2, \Sigma_2)$, $l \ge 1$, and $d_k : \mathrm{pos}(\bar{t} \upharpoonright k) \to l$, $k < \omega$, there exist $\bar{s} \le \bar{t}$ in $\mathsf{PC}_\infty(K_2, \Sigma_2)$ and $l' < l$ such that for each $i < \omega$, if k is such that $s_i \in \Sigma_2(\bar{t} \upharpoonright k)$ and $f \in \mathrm{pos}(\bar{s} \upharpoonright i)$, then $d_k(f) = l'$.

This theorem will be recovered by showing that there is a topological Ramsey space dense in $\mathsf{PC}_\infty(K_2, \Sigma_2)$.

A dense subsets forming a topological Ramsey space

$\mathcal{R}(K_2, \Sigma_2) = \{\bar{s} \in \mathsf{PC}_\infty(K_2, \Sigma_2) : \forall l < \omega, \ |\, \mathbf{val}[t_l] \,| = l + 1\}$,
with its inherited partial ordering.

Thm. [D] $(\mathcal{R}(K_2, \Sigma_2), \le, r)$ is a topological Ramsey space which is dense in $\mathsf{PC}_\infty(K_2, \Sigma_2)$.

Remark. The proof of the pigeonhole again relies on the new product tree Ramsey theorem. The application, though, is slightly different.

The generic filter

Since $\mathcal{R}(K_2, \Sigma_2)$ is a topological Ramsey space, it forces a generic filter $\mathcal{G}$ which is selective for $\mathcal{R}(K_2, \Sigma_2)$, hence has complete combinatorics over $L(\mathbb{R})$ in the presence of a supercompact cardinal.

The generic filter induces an ultrafilter $\mathcal{U}$ on $\mathcal{AR}_1$.

$$\mathcal{AR}_1 = \{(m, n, f) : m < n,\ \mathrm{dom}(f) \subseteq [m, n) \text{ and } \mathrm{ran}(f) \subseteq 2\}$$

This filter induces an ultrafilter on $\mathcal{F}_{\mathbf{H}_2} = \{f : \mathrm{dom}(f) \in \mathrm{FIN}$ and $f : \mathrm{dom}(f) \to 2\}$ generated by possibilities on pure candidates and satisfying the partition theorem of [R/S].

Questions

1. What other creature forcings are essentially (topological) Ramsey spaces? Extend this study to streamline approaches to certain classes of creature forcings.
2. What other forced ultrafilters in the literature, or new ones, have complete combinatorics?
3. What other pigeonhole principles and Ramsey theorems will emerge from this investigation?

References

The work in this talk appears in

[D] *Creature forcing and topological Ramsey spaces*, Topology and Its Applications, special issue in honor of Alan Dow's 60th birthday, 18pp, to appear.

Relevant references

[Di Prisco/Llopis/Todorcevic] *Parametrized partitions of products of finite sets*, Combinatorica, 2004.

[Rosłanowski/Shelah] *Partition theorems from creatures and idempotent ultrafilters*, Annals of Combinatorics, 2013.

[Rosłanowski/Shelah] *Norms on possibilities. I. Forcing with trees and creatures*, Memoires of the AMS, 1999.