Putnam Practice Problems #2 (or: Fun with integers!)  
October 15, 2010  
(in no particular order)

(A) (An easier geometric induction!) If $n$ lines are drawn in the plane ($n \geq 3$) so that no two of the lines are parallel, and no three of the lines meet in a single point, then show that among the regions that the lines partitions the plane into, at least one is a triangle.

1. Given any set $S$ of 2010 points in the plane, show that there exists a line $\ell$ so that 1005 points of $S$ are on one side of $\ell$ and 1005 points of $S$ are on the other side of $\ell$.

2. If the number 2010 is written as a sum of at least two positive integers, what is the maximum possible product of these integers?

3. A sequence $\{a_n\}$ of positive real numbers has the properties that $a_0 = 1$ and $a_{n+2} = 2a_n - a_{n+1}$ for all $n \geq 0$. (Note that $a_1$ is not specified.) Find $a_{2010}$.

4. Find the limit $\lim_{n \to \infty} \frac{1^1 + 2^2 + 3^3 + \ldots + n^n}{n^n}$.

General hint for problems 5-9: modular arithmetic.

5. Find all of the prime numbers (if any) among the integers 101, 10101, 1010101, ... .

6. If $P(x)$ is a nonconstant polynomial with integer coefficients, is it possible that $P(n)$ is prime for all integers $n$?

7. The Fibonacci sequence is defined by $a_0 = 1$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$. Show that some Fibonacci number $a_n$ is divisible by 2010.

8. Find all solutions to $x^2 + 2y^2 + 4z^2 = 0$ where $x, y, z$ are integers.

9. Does there exist an polynomial $p(x)$ with integer coefficients so that $p(1) = 1$, $p(2) = 2$, and $p(3) = 4$?