Putnam Practice Problem Set #4 (Fun with integers!)  
October 28th, 2011  
(in no particular order)

1. Define a sequence of polynomials \( \{ f_n(x) \} \) recursively by \( f_0(x) = 1 \) and \( f_n'(x) = nf_{n-1}(x+1), \quad f_n(0) = 0 \) for \( n \geq 1 \). Find, with proof, the explicit factorization of \( f_{100}(1) \) into powers of distinct primes.

2. Let \( X \) be the number of integers \( n \) in the range \( 1 \leq n \leq 2011 \) such that the first digit of \( 2^n \) equals 1. Prove that \( 2^{2011} \) has exactly \( X \) digits.

3. Prove that in any set of ten consecutive positive integers, one of them is relatively prime to the product of the other nine. (Two integers \( m \) and \( n \) are relatively prime if and only if \( \gcd(m,n) = 1 \).)

4. Find a positive integer \( B \) with the following property: there are exactly 2012 positive integers \( A < B \) such that \( \text{lcm}\{A,B\} + \text{gcd}\{A,B\} = A + B \).

5. Find the number of ways that 2011 can be written in the form \( \sum a_i 2^i \), where the \( a_i \) are allowed to take the values 0, 1, 2, and 3.

6. Let \( P(x) = x^d + a_{d-1}x^{d-1} + a_{d-2}x^{d-2} + \cdots + a_1x + a_0 \) be a monic polynomial with integer coefficients. Suppose that \( P \) has \( d \) positive roots \( r_1, \ldots, r_d \). Show that \( r_1^{2011} + \cdots + r_d^{2011} \geq d \).

7. Show that there is a multiple of \( 2011^{2011} \) which contains all digits 0,1,\ldots,9 in its decimal expansion.

8. A perfect square has tail \( n \) if its last \( n \) digits in base 10 are the same and non-zero. What is the longest possible tail? What is the smallest square with this tail?