Putnam Practice Problem Set #5 (Fun with geometry!)  
November 11th, 2011  
(in no particular order)

(1) Let $A_1, A_2, \ldots , A_n$ be points on a circle of radius 1. Prove that there exists a point $P$ on the circle such that $PA_1 + PA_2 + \cdots + PA_n > n$. (Here, $PA_j$ denotes the distance between the points $P$ and $A_j$.)

(2) Suppose you inscribe regular polygon with 2011 sides inside a circle of radius 1, and then you draw every possible diagonal of the polygon. Evaluate the product of the lengths of all the sides and diagonals of this polygon.

(3) 2011 identical particles are moving along the real axis at constant velocities. You are given snapshots, each of which shows the positions of all the particles at a particular time; in each individual snapshot, the 2011 particles are in 2011 different positions, although you can’t tell which one is which since they’re identical. Show that if you have a snapshot at some initial time and snapshots for 1 second later, 2 seconds later, and so on up to 2011 seconds later, then you can determine the velocities of each of the particles.

(4) What is the maximum possible area of a quadrilateral with three sides of length 1?

(5) Let $C_0$ be the circle centered at $(1, 1)$ with radius 1, and let $C_1$ be the circle centered at $(-1, 1)$ with radius 1. For each $k \geq 2$, let $C_k$ be the circle that lies in the region between $C_0$, $C_{k-1}$, and the $x$-axis and is tangent to all three. For example, $C_2$ is the circle centered at $(0, \frac{1}{4})$ with radius $\frac{1}{4}$. What is the radius of $C_{2011}$?

(6) Let $T$ be a triangle, and let $P$ be a parallelogram that lies inside $T$. (It may intersect $T$ on the boundary.) Show that the area of $P$ is at most one-half the area of $T$.

(7) Let $C_1$ and $C_2$ be two circles in the plane such that the center of $C_1$ lies on $C_2$. Let $W$ and $X$ be the two points of intersection of the circles $C_1$ and $C_2$. Let $L$ be a line through $W$ that is not tangent to either circle, such that no point on the line is inside both circles. Let $Y$ and $Z$ be the points of intersection of $L$ with $C_1$ and $C_2$, respectively. Show that $\triangle XYZ$ is isosceles.

(8) Let $C_1$, $C_2$, and $C_3$ be three circles of radius 1 in the plane, no two of which are tangent to each other, that all pass through a point $P$. Each pair of circles intersects in a second point besides $P$; call these three other points of intersection $Q$, $R$, and $S$. Now draw three new circles $D_1$, $D_2$, and $D_3$ of radius one, centered at the three points $Q$, $R$, and $S$. Show that $D_1$, $D_2$, and $D_3$ all intersect in a common point.