A Parameterized Family of Equilibrium Profiles for Three-Player Kuhn Poker

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ABSTRACT

This paper presents a parameterized family of equilibrium strategy profiles for three-player Kuhn poker. This family illustrates an important feature of three-player equilibrium profiles that is not present in two-player equilibrium profiles - the ability of one player to transfer utility to a second player at the expense of the third player, while playing a strategy in the profile family. This family of strategy profiles was derived analytically and the proof that the members of this family are equilibrium profiles is an analytic one. In addition, the problem of selecting a robust strategy from an equilibrium profile is discussed.

Categories and Subject Descriptors

I.2.1 [**Artificial Intelligence**]: Applications and Expert Systems—*Games*

General Terms

Theory, Economics

Keywords

Game theory; Nash equilibrium; Poker

1. INTRODUCTION

Many problems that involve multiple interacting agents can be modeled as extensive-form games. Poker has been a testbed for studying solution techniques on extensive-form games for over 60 years. The game incorporates two features that occur frequently in multi-agent problems, stochastic events (chance cards) and imperfect information (hidden cards). Poker is complex enough that good solution strategies are usually non-obvious and include multiple types of bluffing, such as acting aggressively with weak hands and slow-playing with strong hands.

In game theory, the most popular solution concept is the Nash equilibrium, where an agent cannot increase utility by varying its strategy unilaterally. The tree sizes of popular poker variants (3×10^{14} information sets for two-player limit Texas Hold'em poker and 5×10^{17} information sets for three-player limit Texas Hold'em) are too large to solve analytically. Instead, insights into the general nature of good strategies are often discovered by studying smaller poker games. Two-player Kuhn poker [9] is arguably the most popular of these small poker games. Although the game is complex enough that the equilibrium solutions support bluffing, the game is small enough that Kuhn discovered all equilibrium solutions analytically. These and other analytic solutions to small games have contributed to the discovery and evaluation of many artificial intelligence techniques. For example, Hoehn *et al.*[7] directly used the Kuhn poker analytical solutions to develop strong exploitive algorithms. In addition, Ganzfried and Sandholm [4] exploited aspects of the equilibrium structure in small games to develop new algorithms for solving certain classes of games and improve endgame performance in two-player limit Texas Hold'em. Other advances have come in the areas of opponent modeling [14], new representations and solution techniques for extensive-form games [8], abstraction techniques [5], and general reinforcement learning algorithms [15].

Recently, Abou Risk and Szafron [1] introduced a three-player version of Kuhn poker to study the performance of the Counterfactual Regret Minimization (CFR) [16] algorithm in games with more than two players. While CFR is guaranteed to converge to a Nash equilibrium in two-player zero-sum games, this guarantee does not hold for general sum games [1, Table 2]. Despite this, Abou Risk and Szafron found CFR to compute approximate Nash equilibria for three-player Kuhn poker, as well as winning strategies in the three-player Texas Hold'em events of the Annual Computer Poker Competition [6]. No insight into the structure or behavior of these profiles, however, was provided. From both an analytical and computational standpoint, two-player games are well understood, while three-player games remain much less understood.

In this paper, we describe a parameterized family of strategy profiles for three-player Kuhn poker and prove that they are Nash equilibria. These profiles exhibit an interesting behavior where one player can transfer utility from one opponent to the other without departing from equilibrium. To our knowledge, this is one of the largest three-player games to be solved analytically to date. The analytically-derived family of equilibrium strategy profiles for two-player Kuhn poker have contributed to our deeper understanding of Nash equilibrium profiles and solvers in two-player zero-sum games. The discovery of parameterized families of equilibrium strategy profiles in three-player Kuhn poker may have a similar impact on research into many-player games. Our hope is that these profiles will be useful to a variety of researchers who are studying a wide range of problems that rely on equilibrium profiles in three or more agent scenarios.

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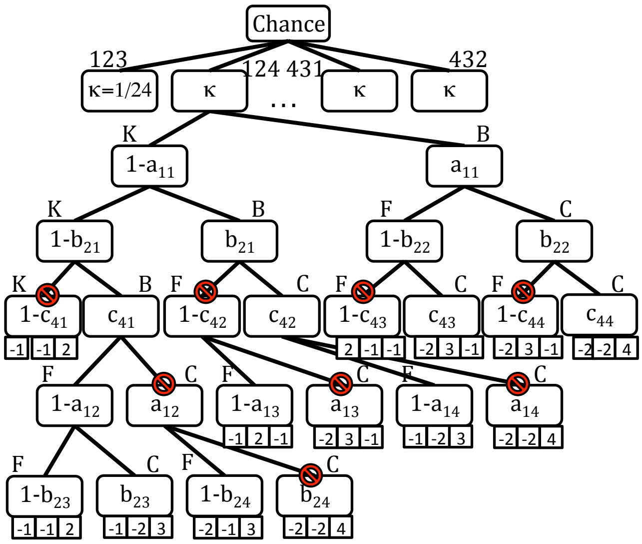

Figure 1: A partial three-player Kuhn poker game tree.

2. THREE-PLAYER KUHN POKER

Three-player Kuhn poker [1] is played with a deck of 4 cards, denoted 1, 2, 3 and 4. To begin, each player must **ante** one chip into the **pot** so that the players have incentive to play. Play then proceeds according to the (partial) game tree shown in Figure 1 (ignore the Ø symbols for now). **Chance** deals each player a hand containing a single card and at the **showdown**, the player with the highest card ($4 > 3 > 2 > 1$) who has not folded wins all the chips in the pot. Each edges to an immediate child node is marked with three digits signifying the card held by each player, P_1, P_2 and P_3 respectively. For brevity, only 4 of the 24 different hand combinations and the sub-nodes of the *124* node are shown.

Each player then acts in turn, starting with P_1, who may **check (*K*)** and put no additional chips in the pot, or **bet (*B*)** and put one additional chip in the pot. If P_1 checked, then P_2 may also check or bet. However, if P_1 bet, then P_2 may either **fold (*F*)** and abandon any chance of winning the pot, or **call (*C*)** and put one additional chip in the pot to match the bet. It is then P_3's turn. If no bet has been made by either P_1 or P_2, then P_3 may check or bet. If a bet was made by either P_1 or P_2, then P_3 may fold or call. If no player bet, the showdown takes place between all three players and the winner wins three chips. If P_1 bet, then there is a showdown between P_1 and any players who called the bet. If no one called, then P_1 wins the pot. If P_1 did not bet and P_2 bet, then after P_3 has acted, P_1 has the opportunity to either fold or call. Then a showdown takes place between P_2 and any callers. The final case is where neither P_1 nor P_2 bet, but P_3 bet. In this case, both P_1 and P_2 can fold or call in order and then the showdown takes place.

Each terminal node of Figure 1 is marked with three integer values. These values are the number of chips won or lost by P_1, P_2 and P_3 respectively. For example, consider the leaf node marked a_{13} (this marking is explained in the following section). Each player contributed 1 chip as an ante so the pot size was 3. Then, tracing the path from the root down to this node, P_1 checked, P_2 bet 1 chip to make the pot size 4 and then P_3 folded. Finally, P_1 contributed 1 chip by calling to make the pot size 5. In the showdown between P_1 and P_2, P_2 wins since the card 2 is greater than the card 1. This means that the utility of P_1 is -2, the utility of P_2 is $5 - 2 = 3$ and the utility of P_3 is -1.

Table 1: Betting situations in three-player Kuhn poker.

Situation	P_1	P_2	P_3
1	-	K	KK
2	KKB	B	KB
3	KBF	$KKBF$	BF
4	KBC	$KKBC$	BC

3. GAME THEORY BACKGROUND

Three-player Kuhn poker is an example of an extensive-form game [10] that contains a game tree with nodes corresponding to **histories** of actions $h \in H$ and edges corresponding to **actions** $a \in A(h)$ available to **player** $P(h) \in N \cup \{\text{Chance}\}$ (where N is the set of players). In three-player Kuhn, we have $N = \{P_1, P_2, P_3\}$ and only one node belongs to Chance (the root). Terminal nodes correspond to **terminal histories** $z \in Z \subseteq H$ that have associated **utilities** $u_i(z)$ for each player P_i. The utilities in three-player Kuhn poker are simply the number of chips won or lost at the end of the game. Non-terminal histories for P_i are partitioned into **information sets** $I \in \mathcal{I}_i$ representing the different game states that P_i cannot distinguish between. Figure 2 shows an example of an information set (dashed box) for P_3, where all histories differing only in the private cards held by the opponents are in the same information set. The action sets $A(h)$ must be identical for all $h \in I$, and we denote this set by $A(I)$. In three-player Kuhn poker, we have $A(I) = \{K, B\}$ or $A(I) = \{F, C\}$ depending on the information set I.

A **(behavioral) strategy** for P_i, σ_i, is a function that maps each information set $I \in \mathcal{I}_i$ to a probability distribution over $A(I)$. A **strategy profile** σ is a collection of strategies $\sigma = (\sigma_1, ..., \sigma_{|N|})$, one for each player. We let σ_{-i} refer to the strategies in σ excluding σ_i, and $u_i(\sigma)$ to be the expected utility for P_i when players play according to σ.

In Figures 1 and 2, a_{jk}, b_{jk}, and c_{jk} denote the action probabilities for players P_1, P_2 and P_3 respectively when holding card j and taking an aggressive action (B or C) in situation k. The situations denote previous betting actions and are summarized in Table 1. For example, c_{42} is the probability of P_3 holding the 4 and taking the aggressive action C after the previous actions were K followed by B. Since each player selects from exactly two actions everywhere in the game tree, the probability of taking the less aggressive action (K or F) is $1 - a_{jk}$, $1 - b_{jk}$ and $1 - c_{jk}$ respectively. As there are exactly four cards and exactly four situations per player, a strategy profile σ in three-player Kuhn poker is fully defined by assigning action probabilities to the 48 independent parameters $\{a_{jk}, b_{jk}, c_{jk} \mid j, k = 1..4\}$. The expected utility for P_i is computed by summing the probabilities of reaching each leaf node times the utility for P_i at that leaf node. For example, from Figure 1, the contribution of the a_{13} leaf node to u_1 would be $-2\kappa(1 - a_{11})b_{21}(1 - c_{42})a_{13}$, where $\kappa = 1/24$ is the probability of each set of hands being dealt.

The most common solution concept in games is the Nash equilibrium first proposed by John Nash [12]. A strategy profile σ is a **Nash equilibrium** if no player can unilaterally deviate from σ to increase their expected utility; *i.e.*,

$$\max_{\sigma_i'} u_i(\sigma_i', \sigma_{-i}) \leq u_i(\sigma) \text{ for all } i = 1, 2, ..., |N|.$$

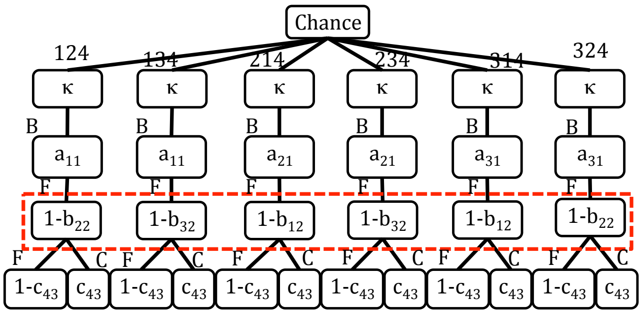

Figure 2: An information set for P_3 in a partial three-player Kuhn game tree.

Table 2: The 21 necessary strategy parameter values in three-player Kuhn poker.

P_1	P_2	P_3
$a_{12} = a_{13} = a_{14} = a_{24} = 0$	$b_{12} = b_{13} = b_{14} = b_{24} = 0$	$c_{12} = c_{13} = c_{14} = c_{24} = 0$
$a_{42} = a_{43} = a_{44} = 1$	$b_{42} = b_{43} = b_{44} = 1$	$c_{42} = c_{43} = c_{44} = 1$

While computing an equilibrium of a two-player zero-sum game can be achieved in polynomial time [8], computing a Nash equilibrium profile in a three-or-more-player game is hard and belongs to the PPAD-complete class of problems [3].

4. NECESSARY PARAMETER VALUES

We now begin our analysis of deriving a family of Nash equilibrium profiles for three-player Kuhn poker. For a strategy parameter x_{jk}, we will denote the information set for which x_{jk} is relevant as $I(x_{jk})$. For example, the information set indicated by the dashed box in Figure 2 is denoted $I(c_{43})$.

In this section, we provide necessary values for 21 of the 48 strategy parameters needed for a strategy profile to be in equilibrium, provided the parameters are "reached:"

DEFINITION 1. *Given a strategy profile σ, x_{jk} is a* ***reached strategy parameter*** *if the probability $\pi^\sigma(I(x_{jk}))$ of reaching its information set $I(x_{jk})$, when the players play according to strategy profile σ, is non-zero. If a strategy parameter is not reached in a strategy profile, it is called a* ***non-reached strategy parameter***.

For example, consider the strategy parameter c_{43} in three-player Kuhn poker. From Figure 2, the probability of reaching $I(c_{43})$ is:

$$\pi^\sigma(I(c_{43})) = \kappa\,[a_{11}(1-b_{22}) + a_{11}(1-b_{32}) + a_{21}(1-b_{12}) + a_{21}(1-b_{32}) + a_{31}(1-b_{12}) + a_{31}(1-b_{22})]\,. \quad (1)$$

In any strategy profile for which this expression is zero, c_{43} is a non-reached strategy parameter. Intuitively, if an information set is never reached when playing according to a strategy profile, then the part of the strategy at that information set is not important, since that part of the strategy will never be used.

Table 2 defines values for 21 strategy parameters that, if reached, must be assigned the corresponding value for a profile to be in equilibrium. Intuitively, these values ensure that trivial, dominated errors are necessarily avoided by an equilibrium profile. For example, $a_{12} = a_{13} = a_{14} = 0$ means that P_1 should never call a bet when holding the lowest card since it is a guaranteed loss for P_1 in a showdown. Also, $a_{24} = 0$ insists that P_1 never calls with the second lowest card when faced with a bet and a call from the opponents, as now at least one opponent is guaranteed to have a higher card in the showdown. Furthermore, $a_{42} = a_{43} = a_{44} = 1$ means that P_1 should always call any bet when holding the highest card since it is a guaranteed win for P_1 in a showdown. Similar arguments hold for P_2 and P_3's necessary strategy parameter values.

THEOREM 1. *P_1 (P_2, P_3) cannot gain utility by unilaterally changing any of the seven a_{jk} (seven b_{jk}, seven c_{jk}) parameters shown in Table 2 from the given values. Every equilibrium strategy profile of three- player Kuhn poker has the parameter values listed in Table 2 unless a parameter is a non-reached strategy parameter in the equilibrium profile.*

The proof of Theorem 1 can be found in the Appendix at the end of the paper. All the nodes that appear in Figure 1, but can never be reached if no player plays a strategy that violates the constraints of Table 2, are marked with a Ø in Figure 1. Naturally, child nodes of a marked node cannot be reached either if no player violates the constraints in Table 2. Note, however, that reached nodes in Figure 1 will differ according to the private cards that are dealt. Theorem 1 leaves 27 independent parameters to consider for equilibrium strategy profiles.

5. A PARAMETERIZED FAMILY OF EQUILIBRIUM PROFILES

Table 2 and Table 3 together, define a family of parameterized equilibrium strategy profiles for three-player Kuhn poker. The parameters listed in Table 3 were discovered through careful examination of a set of solutions produced by a Monte Carlo sampling variant of Counterfactual Regret Minimization [11] with different random seeds. There are actually three sub-families of equilibrium profiles defined by: $c_{11} = 0$, $c_{11} = 1/2$ and $0 < c_{11} < 1/2$.

The $c_{11} = 0$ sub-family is the simplest of the three sub-families. There are four independent parameters, b_{11}, b_{21}, b_{32} and c_{33}, so the parameter space is a 4-dimensional convex volume. The ranges of the four independent parameters are: $b_{21} \leq 1/4$, $b_{11} \leq b_{21}$, $b_{32} \leq (2 + 3b_{11} + 4b_{21})/4$, and $1/2 - b_{32} \leq c_{33} \leq 1/2 - b_{32} + (3b_{11} + 4b_{21})/4$. The dependent parameters have values, $b_{23} = 0$, $b_{33} = (1 + b_{11} + 2b_{21})/2$, $b_{41} = 2b_{11} + 2b_{21}$, and $c_{21} = 1/2$. As shown in Table 3, the utilities depend only on $\beta = b_{21}$. Figure 3 shows a graphical representation for the domains of b_{11}, b_{21} and b_{32}. The valid domains form the interior of the volume. Figure 4 shows a graphical representation for the domains of b_{11}, b_{21} and c_{33} for three values of b_{32}. For $b_{32} = 7/8$, the domains form a very small polyhedron with c_{33} in the interval $[0, 1/16 = 0.0625]$ when $b_{11} = b_{21} = 1/4$. For the largest value of $b_{32} = 15/16$, the polyhedron reduces to a single point, $b_{11} = b_{21} = 1/4$, $c_{33} = 0$ (not shown in Figure 4).

The $0 < c_{11} < 1/2$ sub-family is the next simplest. There are four independent parameters, b_{11}, b_{32}, c_{11} and c_{33}, so the parameter space is a convex 4-dimensional volume. The ranges of the four independent parameters are: $b_{11} \leq 1/4$, $b_{32} \leq (2 + 7b_{11})/4$, $c_{11} < 1/2$ when $b_{11} \leq 1/6$ and $c_{11} \leq (2 - b_{11})/(3 + 4b_{11})$ when $1/6 < b_{11} \leq 1/4$, and $1/2 - b_{32} \leq c_{33} \leq 1/2 - b_{32} + 7b_{11}/4$. The dependent parameters have values, $b_{21} = b_{11}$, $b_{23} = 0$,

Table 3: Parameter values and utilities for a three-player Kuhn family of equilibrium profiles, where $\beta = \max\{b_{11}, b_{21}\}$ and $\kappa = 1/24$ is the probability of each set of hands.

P_1	P_2	P_3
$a_{11} = 0$	$b_{11} \leq b_{21}$ if $c_{11} = 0$ $b_{11} \leq \frac{1}{4}$ if $c_{11} \neq 0$	$c_{11} \leq \min\{\frac{1}{2}, (2-b_{11})/(3+2b_{11}+2b_{21})\}$
$a_{21} = 0$	$b_{21} \leq \frac{1}{4}$ if $c_{11} = 0$ $b_{21} = b_{11}$ if $0 < c_{11} < \frac{1}{2}$ $b_{21} \leq \min\{b_{11}, \frac{1}{2} - 2b_{11}\}$ if $c_{11} = \frac{1}{2}$	$c_{21} = \frac{1}{2} - c_{11}$
$a_{22} = 0$	$b_{22} = 0$	$c_{22} = 0$
$a_{23} = 0$	$b_{23} \leq \max\{0, (b_{11} - b_{21})/2(1 - b_{21})\}$	$c_{23} = 0$
$a_{31} = 0$	$b_{31} = 0$	$c_{31} = 0$
$a_{32} = 0$	$b_{32} \leq \frac{1}{2} + \frac{3}{4}(b_{11} + b_{21}) + \frac{\beta}{4}$	$c_{32} = 0$
$a_{33} = \frac{1}{2}$	$b_{33} = \frac{1}{2} + \frac{1}{2}(b_{11} + b_{21}) + \frac{\beta}{2} - b_{23}(1-b_{21})$	$\frac{1}{2} - b_{32} \leq c_{33} \leq \frac{1}{2} - b_{32} + \frac{3}{4}(b_{11}+b_{21}) + \frac{\beta}{4}$
$a_{34} = 0$	$b_{34} = 0$	$0 \leq c_{34} \leq 1$
$a_{41} = 0$	$b_{41} = 2b_{11} + 2b_{21}$	$c_{41} = 1$
$u_1 = -\kappa(\frac{1}{2} + \beta)$	$u_2 = -\kappa(\frac{1}{2})$	$u_3 = \kappa(1 + \beta)$

$b_{33} = (1 + 3b_{11})/2$, $b_{41} = 4b_{11}$, and $c_{21} = 1/2 - c_{11}$. The utilities depend only on $\beta = b_{11}$. Figure 5 shows a graphical representation for the domains of b_{11}, b_{32} and c_{33}. Figure 6 shows the b_{11} and c_{11} domains.

The $c_{11} = 1/2$ sub-family is the most complex. There are five independent parameters, b_{11}, b_{21}, b_{23}, b_{32} and c_{33}, so the parameter space is a convex 5-dimensional volume. The ranges of the five independent parameters are: $b_{11} \leq 1/4$, $b_{21} \leq b_{11}$ when $b_{11} \leq 1/6$ and $b_{21} \leq 1/2 - 2b_{11}$ when $1/6 \leq b_{11} \leq 1/4$, $b_{23} \leq (b_{11} - b_{21})/2(1 - b_{21})$, $b_{32} \leq (2 + 4b_{11} + 3b_{21})/4$, and $1/2 - b_{32} \leq c_{33} \leq 1/2 - b_{32} + (4b_{11} + 3b_{21})/4$. The dependent parameters have values $b_{33} = (1 + b_{11} + 2b_{21})/2$, $b_{41} = 2b_{11} + 2b_{21}$, and $c_{21} = 0$. The utilities depend only on $\beta = b_{11}$. Figure 7 shows the domains of b_{11}, b_{21}, b_{23}, and b_{32}. By symmetry, Figure 4 shows the domains of b_{11}, b_{21}, and c_{33} for three values of b_{32}, except that the b_{11} and b_{21} axes in Figure 4 must be swapped to switch from the $c_{11} = 0$ sub-family to the $c_{11} = 1/2$ sub-family.

All of these equilibrium strategy profiles share some common features. First, the utility of P_2 is fixed and negative. Second the utility of P_1 is always less than or equal to P_2's utility. Third, P_2 completely controls an amount of utility, $\kappa\beta$, that can be transferred from P_1 to P_3 by changing a single strategy parameter. If P_3 plays the $c_{11} = 0$ strategy, the parameter is b_{21}. Otherwise, it is b_{11}. In either case, P_2 can transfer the maximum utility by playing $b_{11} = b_{21} = 1/4$ and transfer the minimum utility by playing $b_{11} = b_{21} = 0$. In the entire family of equilibrium profiles, P_1 always

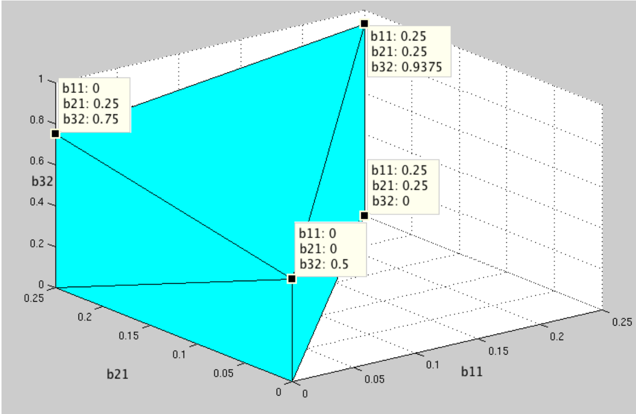

Figure 3: Sub-family $c_{11} = 0$ domains for parameters b_{11}, b_{21} and b_{32}.

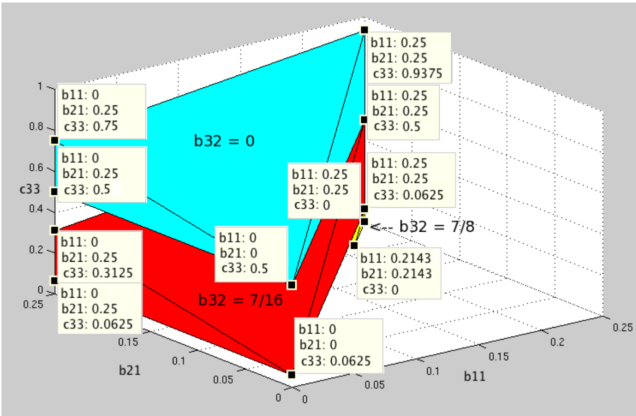

Figure 4: Sub-family $c_{11} = 0$ domains for parameters b_{11}, b_{21} and c_{33} for three parameter values in $0 \leq b_{32} \leq 15/16$.

checks as the first action ($a_{11} = a_{21} = a_{31} = a_{41} = 0$). If P_2 then bets with a weak card (1 or 2), the uncertainty translates to a positive outcome for P_3 at the expense of P_1.

We summarize the main result of this paper in the following theorem. The proof is in the Appendix.

THEOREM 2. *The strategy profiles defined by the parameter values and constraints shown in Table 2 and Table 3 comprise equilibrium strategy profiles for three-player Kuhn poker.*

6. EQUILIBRIUM SELECTION

While in two-person zero-sum games equilibrium strategies are fully interchangeable without changing the payoff of the game, this is not the case in multi-player and non-zero-sum games [12]. If all players switch from one equilibrium profile to another, the payoff for each player can change. If players are playing strategies from different equilibrium profiles, the combined strategies do not necessarily form an equilibrium profile. This problem arises in the three-player Kuhn poker equilibrium profiles that we derived. For example, consider the three-player Kuhn non-equilibrium strategy profile defined by the parameter values listed in Table 2 and Table 4. Here, P_2 has chosen a strategy from the $c_{11} = 1/2$ sub-family of Table 3. However, P_3 is playing a $c_{11} = 0$ strategy. This combination of strategies results in a profile where P_2's utility is

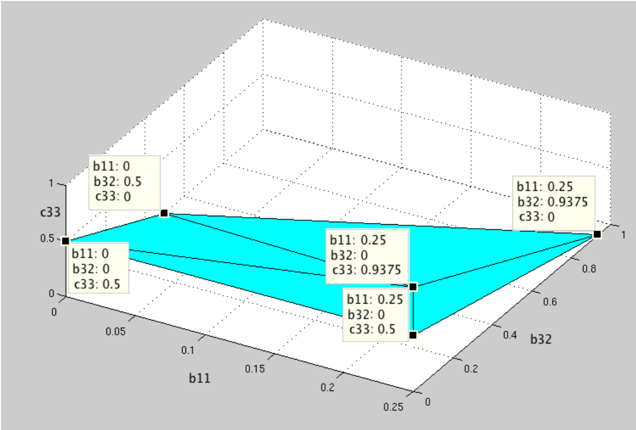

Figure 5: The $0 < c_{11} < 1/2$ sub-family domains of parameters b_{11}, b_{32} and c_{33}.

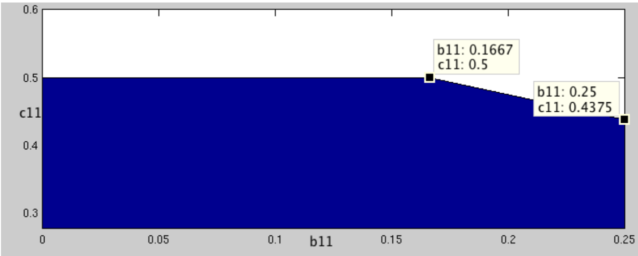

Figure 6: The $0 < c_{11} < 1/2$ sub-family domains of parameters b_{11} and c_{11}.

$-\kappa(1/2+\beta)$, which is $\kappa\beta$ less than what P_2 earns in the equilibrium profile families. Note that this resulting profile is not a Nash equilibrium since P_2 can unilaterally deviate to a strategy in the $c_{11}=0$ sub-family and gain $\kappa\beta$ in utility.

This example shows that some strategies from equilibrium profiles may not guarantee their equilibrium value, even when the other players are also following strategies from other equilibrium profiles. However, some strategies from Table 3 are more robust than others. We now prescribe strategies for each player that guarantee the best worst-case payoff, assuming that all players must play some strategy listed in Table 2 and Table 3.

First off, P_1 has no free parameters in Table 3, and so P_1's strategy σ_1 is fixed. Recall that P_1 never bets with any card to begin the game as $a_{11}=a_{21}=a_{31}=a_{41}=0$. This, in particular, makes c_{33} a non-reached strategy parameter, and thus P_3's choice for c_{33} is irrelevant when computing players' utilities.

Secondly, for P_2, since every profile in Table 3 gives $u_2 = -\kappa/2$, P_2's best worst-case payoff is at most $-\kappa/2$. P_2 can guarantee this payoff by picking any strategy σ_2 from Table 3 where $b_{11}=b_{21}$. This is because from (4), the partial derivative of u_2 with respect to P_3's remaining free parameter, c_{11}, is $\partial u_2/\partial c_{11} = \kappa[8b_{23}(1-b_{21})]/2 > 0$, and thus u_2 is minimized when P_3 chooses any strategy σ_3 from Table 3 with $c_{11}=0$. However, one can easily check that $(\sigma_1,\sigma_2,\sigma_3)$ is an equilibrium profile from Table 3 (except possibly at the non-reached parameter c_{33}), and so $u_2=-\kappa/2$.

Thirdly, P_3 can guarantee a best worst-case payoff of at most κ. This is because any equilibrium profile from Table 3 where P_2

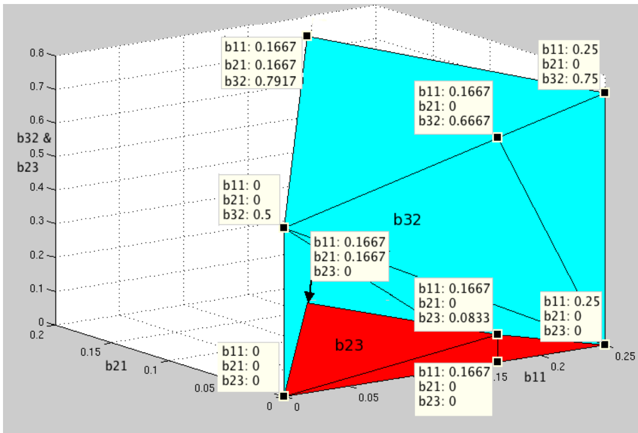

Figure 7: The $c_{11} = 1/2$ sub-family domains of parameters b_{11}, b_{21}, b_{23} and b_{32}.

Table 4: Parameter values and utilities for a three-player Kuhn non-equilibrium strategy profile, where each player is playing a strategy from a different equilibrium profile. Again, $\kappa = 1/24$ is the probability of each set of hands.

P_1	P_2	P_3
$a_{11}=0$	$\beta=b_{11}=\frac{1}{4}$	$c_{11}=0$
$a_{21}=0$	$b_{21}=0$	$c_{21}=\frac{1}{2}$
$a_{22}=0$	$b_{22}=0$	$c_{22}=0$
$a_{23}=0$	$b_{23}=\frac{1}{8}$	$c_{23}=0$
$a_{31}=0$	$b_{31}=0$	$c_{31}=0$
$a_{32}=0$	$b_{32}=0$	$c_{32}=0$
$a_{33}=\frac{1}{2}$	$b_{33}=\frac{5}{8}$	$c_{33}=\frac{1}{4}$
$a_{34}=0$	$b_{34}=0$	$c_{34}=0$
$a_{41}=0$	$b_{41}=\frac{1}{2}$	$c_{41}=1$
$\boldsymbol{u_1=-\kappa(\frac{1}{2})}$	$\boldsymbol{u_2=-\kappa(\frac{1}{2}+\beta)}$	$\boldsymbol{u_3=\kappa(1+\beta)}$

chooses $\beta=0$ gives $u_3=\kappa$. P_3 can guarantee this payoff by picking any strategy σ_3 from Table 3 where $c_{11}=0$. This is because the partial derivatives of u_3 with respect to each of P_2's free parameters are all at least zero, as one can easily check from (3). Thus, P_2 minimizes P_3's payoff by selecting the strategy σ_2' from Table 3 where $b_{11}=b_{21}=b_{23}=b_{32}=0$. Finally, $(\sigma_1,\sigma_2',\sigma_3)$ is an equilibrium profile from Table 3 (except possibly at c_{33}) with $\beta=0$, and so $u_3=\kappa$.

In summary, when all players are playing some strategy from Table 2 and Table 3, to guarantee the best worst-case payoffs, P_1's parameters are fixed, P_2 should choose any strategy with $b_{11}=b_{21}$ and P_3 should choose $c_{11}=0$.

7. RELATED WORK

As one might expect, the parameterized family of equilibrium profiles for three-player Kuhn poker found here are quite complex compared to those for two-player Kuhn poker. In contrast, the two-player family of equilibrium profiles have just one free parameter $\gamma \in [0,1]$ that defines the probability of the first player betting with the highest card. The choice of this parameter then fixes the first player's other action probabilities. Much like P_1 in three-player Kuhn poker, the second player has no free strategy parameters.

We are not the first to solve a three-player game analytically. Nash and Shapley [13] find equilibrium profiles (which they call *equilibrium points*) for a three-player game where there are two kinds of cards *High* and *Low* in the deck. However, the game differs in two fundamental ways from real poker games, one is that the cards are sampled with replacement and the other is that the antes are returned if no one bets. The first factor is significant since it reduces the inferences that can be made by the players about what cards the opponents hold. In addition, Chen and Ankenman [2, Example 29.2] construct an end-game scenario of a three-player poker game and discuss the equilibrium profiles that arise. In fact, their example was construed so that one player could transfer utility between the two opponents while remaining in equilibrium, much like P_2 can in the three-player Kuhn profiles that we derive. Our results show that this interesting behavior can arise naturally and unintentionally in real games.

8. CONCLUSION

This paper has analytically derived a family of Nash equilibrium profiles for three-player Kuhn poker. To our knowledge, this is the largest game with more than two players to be solved analytically to date. It remains open as to whether there exist other three-player Kuhn equilibrium profiles that do not belong to this family.

The equilibrium profiles exhibit an interesting property where P_2 can shift utility between P_1 and P_3 by adjusting a free parameter β, all while staying in equilibrium. We hope that the profiles presented here for three-player Kuhn poker provide future insights into behaviors in other environments involving more than two agents. In addition, these analytical solutions should enable research about learning in two-player Kuhn poker to be extended to three players.

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Appendix

In this appendix, we prove Theorems 1 and 2. Much of the proofs make repeated use of the following lemma:

LEMMA 1. *Let x be a, b or c, when $i = 1, 2$ or 3 respectively. Then P_i cannot gain utility by unilaterally changing the value of x_{jk} if and only if one of the following is true: 1) $x_{jk} = 0$ and $\partial u_i/\partial x_{jk} \leq 0$, or 2) $x_{jk} = 1$ and $\partial u_i/\partial x_{jk} \geq 0$ or 3) $0 < x_{jk} < 1$ and $\partial u_i/\partial x_{jk} = 0$.*

PROOF. Since u_i is linear in each x_{jk}, $u_i = yx_{jk} + z$ for some y, z. Also, x_{jk} is a probability so either 1) $x_{jk} = 0$, or 2) $x_{jk} = 1$ or 3) $0 < x_{jk} < 1$.

Case 1) Assume $x_{jk} = 0$.

(Forward proof): Assume $\partial u_i/\partial x_{jk} \leq 0$, then $y \leq 0$. Assume P_i unilaterally changes the value of x_{jk} so that $x'_{jk} > 0$. Then, the new utility, $u'_i = yx'_{jk} + z \leq y(0) + z$ (since $y \leq 0$ and $x'_{jk} > 0$) = $yx_{jk} + z$ (since $x_{jk} = 0$) = u_i. Therefore, when P_i unilaterally changes the value of x_{jk} from zero, the new utility cannot increase.

(Backward proof): Instead assume that P_i cannot gain utility by unilaterally changing the value of x_{jk}. Then $y \leq 0$ so $\partial u_i/\partial x_{jk} \leq 0$.

The proofs for the other two cases are similar. □

PROOF. (of Theorem 1). Here is a proof for c_{43}. The other 20 proofs are similar.

$$\begin{aligned}\partial u_3/\partial c_{43} = 4\kappa\,[&a_{11}(1-b_{22}) + a_{11}(1-b_{32}) + a_{21}(1-b_{12}) \\ &+a_{21}(1-b_{32}) + a_{31}(1-b_{12}) + a_{31}(1-b_{22})] \geq 0.\end{aligned} \quad (2)$$

(Forward direction): First assume $c_{43} = 1$, then from (2), P_3 cannot gain utility by unilaterally changing c_{43}.

(Backward direction): Instead assume P_3 cannot gain utility by unilaterally changing c_{43}. Assume $c_{43} \neq 1$. Then by Lemma 1 $\partial u_3/\partial c_{43} \leq 0$, but we have shown $\partial u_3/\partial c_{43} \geq 0$. Therefore $\partial u_3/\partial c_{43} = 0$. However, from (1) and (2), $\partial u_3/\partial c_{43} =$

$4\pi^{\sigma}(I(c_{43}))$. Therefore either $c_{43} = 1$ by contradiction or c_{43} is a non-reached parameter. $\square$

We will now prove Theorem 2 according to the following outline. We first compute the general utilities of each player u_1, u_2 and u_3 from the extensive-form game tree as a function of the 48 independent parameters. The expressions for u_1, u_2 and u_3 contain 333, 457 and 457 terms respectively. We then use Table 2 to set 21 of these parameters, leaving 27 independent parameters. From Theorem 1, P_i (for $i = 1, 2, 3$) cannot gain utility by changing the appropriate values from Table 2. The expressions for u_1, u_2 and u_3 now contain 134, 158 and 167 terms respectively. Next, we set the parameters in Table 3 for P_1 and P_2, but do not set parameters for P_3 from Table 3. The utility for P_3 is:

$$\begin{aligned} u_3 = \kappa\,[&4c_{11}b_{21}b_{23} - c_{31}b_{21}b_{23} - b_{11} - b_{21} - 4c_{31} + 2c_{21} \\ &+ 2c_{11} + 2b_{23}c_{41} + 2b_{33}c_{41} - b_{21}c_{32} - b_{11}c_{32} - b_{11}c_{22} \\ &- 4c_{21}b_{33} - 4c_{11}b_{33} + c_{31}b_{23} - 4c_{11}b_{23} + 5b_{21}c_{31} \\ &+ 2b_{11}c_{21} + 4b_{21}c_{21} + 5b_{11}c_{31} + 4b_{11}c_{11} + 2b_{21}c_{11} \\ &-4b_{21}c_{22} - 2b_{21}b_{23}c_{41}]\,. \end{aligned} \tag{3}$$

For the family of strategy profiles defined by Table 2 and Table 3 to be an equilibrium strategy profile, it is necessary to show that changing the parameter values of P_3 from the values listed in Table 3 cannot increase u_3. Similarly it is also necessary to show that the value of u_2 computed by setting the parameters for P_1 and P_3 from Table 3 (but not P_2's parameters) cannot be increased by changing the values of P_2 from the values listed in Table 3, where:

$$\begin{aligned} u_2 = \frac{\kappa}{2}\,[&4b_{21}b_{23} + b_{31}b_{34} - 1 - 5b_{31} - 4b_{23} - b_{34} \\ &-8c_{11}b_{21}b_{23} + 8c_{11}b_{23}]\,. \end{aligned} \tag{4}$$

Finally, it is necessary to show the analogous result P_1, where

$$\begin{aligned} u_1 = \frac{\kappa}{2}\,[&-1 - 4a_{31}b_{21}a_{32} + 8b_{11}c_{11}a_{32} + 6a_{31}b_{11}c_{11} - 2a_{41} \\ &+ 4a_{11} - 4a_{22} + 4a_{21} + 4a_{41}b_{32} - 8a_{11}b_{32} - 8a_{21}b_{32} \\ &- 6a_{31}b_{21}c_{11} - 8a_{11}c_{33} - 8a_{21}c_{33} + 4c_{11}b_{11}a_{22} \\ &- 2b_{21}a_{34} - 2b_{11}a_{21}a_{22} + 8a_{21}b_{21}a_{23} + 4a_{41}c_{33} - 2b_{21} \\ &- 5a_{31} - a_{32} + 2b_{11}a_{31}a_{34} + 2b_{21}a_{31}a_{34} - 8b_{21}a_{23} \\ &- 3b_{11}a_{41} + a_{31}a_{32} + 2b_{11}a_{22} + 4b_{21}a_{32} - 4b_{21}a_{41} \\ &+ 6c_{11}a_{22} - 6c_{11}a_{22}a_{21} + 2c_{11}b_{21}a_{41} - 2b_{11}a_{41}c_{11} \\ &- 4c_{11}b_{11}a_{21}a_{22} - 4c_{11}a_{21}b_{21}a_{22} - 2b_{11}a_{34} - 4b_{11}c_{11} \\ &+ 4b_{21}c_{11} + 4a_{21}a_{22} - 8a_{31}b_{11}c_{11}a_{32} + 8a_{31}b_{21}c_{11}a_{32} \\ &-8b_{21}c_{11}a_{32} + 4c_{11}b_{21}a_{22} + 3a_{31}b_{11} + 6a_{31}b_{21}]\,. \end{aligned} \tag{5}$$

However, showing all these 3 necessary conditions is also sufficient based on the definition of equilibrium profiles. To prove Theorem 2, we divide the proof into four lemmas.

LEMMA 2. *The P_2 constraints in Table 3 imply that $b_{11} \le 1/4$ and $b_{21} \le 1/4$.*

PROOF. One can easily verify the lemma by checking all three cases for $c_{11} = 0$, $0 < c_{11} < 1/2$, and $c_{11} = 1/2$. $\square$

LEMMA 3. *P_3 cannot increase u_3 by changing the parameter values listed in Table 2 and Table 3.*

PROOF. We use (3) to compute the partial derivative of u_3 with respect to each of the listed P_3 parameters. Recall that all parameters are probabilities in the range $[0, 1]$.

$$\partial u_3/\partial c_{23}, \partial u_3/\partial c_{33}, \partial u_3/\partial c_{34} = 0$$

$\Rightarrow$ P_3 cannot change c_{23}, c_{33} or c_{34} to increase u_3.

$$\partial u_3/\partial c_{22} = \kappa(-b_{11} - 4b_{21}) \le 0$$

$\Rightarrow$ P_3 cannot change c_{22} from 0 to increase u_3.

$$\partial u_3/\partial c_{32} = \kappa(-b_{11} - b_{21}) \le 0$$

$\Rightarrow$ P_3 cannot change c_{32} from 0 to increase u_3.

$$\begin{aligned} \partial u_3/\partial c_{31} &= \kappa(-4 + 5b_{11} + 5b_{21} + b_{23} - b_{21}b_{23}) \\ &\le \kappa(-4 + 5/4 + 5/4 + 1 - b_{21}b_{23}) \text{ by Lemma 2} \\ &= \kappa(-1/2 - b_{21}b_{23}) < 0 \end{aligned}$$

$\Rightarrow$ P_3 cannot change c_{31} from 0 to increase u_3.

$$\partial u_3/\partial c_{41} = \kappa(2b_{33} + 2b_{23}(1 - 2b_{21})) \ge 0$$

$\Rightarrow$ P_3 cannot change c_{41} from 1 to increase u_3.

$$\begin{aligned} \partial u_3/\partial c_{21} &= \kappa(2 + 2b_{11} + 4b_{21} - 4b_{33}) \\ &= \kappa[2 + 2b_{11} + 4b_{21} - 4(1/2 + (b_{11} + b_{21})/2 \\ &\quad + \beta/2 - b_{23}(1 - b_{21}))] \text{ by Table 3 } b_{33} \text{ constraint} \\ &= \kappa(2b_{21} - 2\beta + 4b_{23}(1 - b_{21})). \end{aligned}$$

$$\begin{aligned} \partial u_{33}/\partial c_{11} &= \kappa(2 + 4b_{11} + 2b_{21} - 4b_{23}(1 - b_{21}) - 4b_{33}) \\ &= \kappa[2 + 4b_{11} + 2b_{21} - 4b_{23}(1 - b_{21}) - 4(1/2 \\ &\quad + (b_{11} + b_{21})/2 + \beta/2 - b_{23}(1 - b_{21}))] \\ &\quad \text{by Table 3 } b_{33} \text{ constraint} \\ &= \kappa(2b_{11} - 2\beta). \end{aligned}$$

The rest of the proof depends on the family constraints.

If $c_{11} = 0$, then from the b_{11} entry of Table 3, $b_{11} \le b_{21}$, so the b_{23} entry of Table 3 yields $b_{23} = 0$ and $\beta = b_{21}$. Therefore,

$$\partial u_3/\partial c_{21} = \kappa(2b_{21} - 2b_{21} + 4(0)(1 - b_{21})) = 0$$

$\Rightarrow$ P_3 cannot change c_{21} to increase u_3.

$$\partial u_3/\partial c_{11} = \kappa(2b_{11} - 2b_{21}) \le 0$$

$\Rightarrow$ P_3 cannot change c_{11} from 0 to increase u_3.

If $c_{11} = 1/2$, then from the b_{21} entry of Table 3, $b_{21} \le b_{11}$, so the b_{23} entry of Table 3 yields $b_{23} \le (b_{11} - b_{21})/(2(1 - b_{21}))$ and $\beta = b_{11}$. Therefore,

$$\partial u_3/\partial c_{21} \le \kappa(2b_{21} - 2b_{11} + 4(1/2)(b_{11} - b_{21})) = 0$$

$\Rightarrow$ P_3 cannot change c_{21} from 0 to increase u_3.

$$\partial u_3/\partial c_{11} = \kappa(2b_{11} - 2b_{11}) = 0$$

$\Rightarrow$ P_3 cannot change c_{11} to increase u_3.

If $0 < c_{11} < 1/2$, then from the b_{21} entry of Table 3, $b_{21} = b_{11}$, so the b_{23} entry of Table 3 yields $b_{23} = 0$ and $\beta = b_{11} = b_{21}$. Therefore,

$$\partial u_3/\partial c_{21} = \kappa(2b_{21} - 2b_{21} + 4(0)(1 - b_{21})) = 0$$

$\Rightarrow$ P_3 cannot change c_{21} to increase u_3.

$$\partial u_3/\partial c_{11} = \kappa(2b_{11} - 2b_{11}) = 0$$

$\Rightarrow$ P_3 cannot change c_{11} to increase u_3. $\square$

LEMMA 4. *P_2 cannot increase u_2 by changing the parameter values listed in Table 2 and Table 3.*

PROOF. We use (4) to compute the partial derivative of u_2 with respect to each of the listed P_2 parameters.

$$\partial u_2/\partial b_{11}, \partial u_2/\partial b_{22}, \partial u_2/\partial b_{32}, \partial u_2/\partial b_{33}, \partial u_2/\partial b_{41} = 0$$

$\Rightarrow P_2$ cannot change $b_{11}, b_{22}, b_{32}, b_{33}$ or b_{41} to increase u_2.

$$\partial u_2/\partial b_{31} = \kappa(-5 + b_{34})/2 < 0$$

$\Rightarrow P_2$ cannot change b_{31} from 0 to increase u_2.

$$\partial u_2/\partial b_{34} = \kappa(-1 + b_{31})/2 \leq 0$$

$\Rightarrow P_2$ cannot change b_{34} from 0 to increase u_2.

$$\partial u_2/\partial b_{23} = 2\kappa(1 - b_{21})(1 - 2c_{11}).$$

$$\partial u_2/\partial b_{21} = 2\kappa b_{23}(1 - 2c_{11}).$$

The rest of the proof depends on the sub-family constraints.

If $c_{11} = 0$, then

$$\partial u_2/\partial b_{23} = -2\kappa(1 - b_{21}) \leq 0$$

$\Rightarrow P_2$ cannot change b_{23} from 0 to increase u_2. Therefore, we can assume $b_{23} = 0$.

$$\partial u_2/\partial b_{21} = 2\kappa b_{23} = 0$$

$\Rightarrow P_2$ cannot change b_{21} to increase u_2.

If $c_{11} = 1/2$, then

$$\partial u_2/\partial b_{23} = \partial u_2/\partial b_{21} = 0$$

$\Rightarrow P_2$ cannot change b_{23} or b_{21} to increase u_2.

If $0 < c_{11} < 1/2$, then

$$\partial u_2/\partial b_{23} = -2\kappa(1 - b_{21})(1 - 2c_{11}) \leq 0$$

$\Rightarrow P_2$ cannot change b_{23} from 0 to increase u_2. Therefore, we can assume $b_{23} = 0$.

$$\partial u_2/\partial b_{21} = 2\kappa(0)(1 - 2c_{11}) = 0$$

$\Rightarrow P_2$ cannot change b_{21} to increase u_2. □

LEMMA 5. *P_1 cannot increase u_1 by changing the parameter values listed in Table 2 and Table 3.*

PROOF. We use (5) to compute the partial derivative of u_1 with respect to each of the P_1 parameters.

$$\partial u_1/\partial a_{33} = 0$$

$\Rightarrow P_1$ cannot change a_{33} to increase u_1.

$$\partial u_1/\partial a_{23} = 4\kappa(-1 + a_{21})b_{21} \leq 0$$

$\Rightarrow P_1$ cannot change a_{23} from 0 to increase u_1. Assume $a_{23} = 0$.

$$\partial u_1/\partial a_{34} = \kappa(-1 + a_{31})(b_{11} + b_{21}) \leq 0$$

$\Rightarrow P_1$ cannot change a_{34} from 0 to increase u_1.

$$\partial u_1/\partial a_{11} = \kappa(2 - 4b_{32} - 4c_{33}) \leq 0$$

by Table 3 constraint $1/2 - b_{32} \leq c_{33} \Rightarrow P_1$ cannot change a_{11} from 0 to increase u_1.

$$\partial u_1/\partial a_{22} = \kappa(1 - a_{21})(b_{11} + 2(b_{11} + b_{21})c_{11} + 3c_{11} - 2) \leq 0$$

by Table 3 constraint $c_{11} \leq (2 - b_{11})/(3 + 2(b_{11} + b_{21})) \Rightarrow P_1$ cannot change a_{22} from 0 to increase u_1. So, assume $a_{22} = 0$.

$$\begin{aligned}\partial u_1/\partial a_{21} &= \kappa(2 - 4b_{32} - 4c_{33} + 2a_{22} + 4a_{23}b_{21} - a_{22}b_{11} \\ &\quad - 2a_{22}(b_{11} + b_{21})c_{11} - 3a_{22}c_{11}) \\ &= \kappa(2 - 4b_{32} - 4c_{33}) \text{ since } a_{22} = a_{23} = 0 \\ &\leq 0 \text{ by Table 3 constraint } 1/2 - b_{32} \leq c_{33}\end{aligned}$$

$\Rightarrow P_1$ cannot change a_{21} from 0 to increase u_1. Therefore, we can assume $a_{21} = 0$.

$$\partial u_1/\partial a_{32} = \kappa(1 - a_{31})(-1 + 4b_{21} + 8c_{11}(b_{11} - b_{21}))/2.$$

$$\begin{aligned}\partial u_1/\partial a_{41} &= \kappa(-2c_{11}(b_{11} - b_{21}) - 2 - 4b_{21} - 3b_{11} + 4b_{32} \\ &\quad + 4c_{33})/2 \\ &\leq \kappa[-2c_{11}(b_{11} - b_{21}) - 2 - 4b_{21} - 3b_{11} + 4b_{32} \\ &\quad + (2 - 4b_{32} + 3(b_{11} + b_{21}) + \beta)]/2 \text{ by Table 3} \\ &\quad \text{constraint } c_{33} \leq 1/2 - b_{32} + 3(b_{11} + b_{21})/4 + \beta/4 \\ &= \kappa(-2c_{11}(b_{11} - b_{21}) - b_{21} + \beta)/2.\end{aligned}$$

The next part of the proof depends on the sub-family constraints.

If $c_{11} = 0$, then

$$\partial u_1/\partial a_{32} = \kappa(1 - a_{31})(-1 + 4b_{21} + 8(0)(b_{11} - b_{21}))/2 \leq 0$$

since $b_{21} \leq 1/4$ by Lemma 2 $\Rightarrow P_1$ cannot change a_{32} from 0 to increase u_1. From the b_{11} entry of Table 3, $b_{11} \leq b_{21}$, so $\beta = b_{21}$.

$$\partial u_1/\partial a_{41} \leq \kappa(-2(0)(b_{11} - b_{21}) - b_{21} + b_{21})/2 = 0$$

$\Rightarrow P_1$ cannot change a_{41} from 0 to increase u_1.

If $c_{11} = 1/2$, then

$$\begin{aligned}\partial u_1/\partial a_{32} &= \kappa(1 - a_{31})(-1 + 4b_{21} + 8(1/2)(b_{11} - b_{21}))/2 \\ &= \kappa(1 - a_{31})(-1 + 4b_{11})/2 \\ &\leq 0 \text{ since } b_{11} \leq 1/4 \text{ from Lemma 2}\end{aligned}$$

$\Rightarrow P_1$ cannot change a_{32} from 0 to increase u_1. From the b_{21} entry of Table 3, $b_{21} \leq b_{11}$, so $\beta = b_{11}$. Therefore,

$$\partial u_1/\partial a_{41} \leq \kappa(-2(1/2)(b_{11} - b_{21}) - b_{21} + b_{11})/2 = 0$$

$\Rightarrow P_1$ cannot change a_{41} from 0 to increase u_1.

If $0 < c_{11} < 1/2$, then from the b_{21} entry of Table 3, $b_{21} = b_{11}$. Therefore,

$$\partial u_1/\partial a_{32} = \kappa(1 - a_{31})(-1 + 4b_{21})/2 \leq 0$$

since $b_{21} \leq 1/4$ by Lemma 2 $\Rightarrow P_1$ cannot change a_{32} from 0 to increase u_1. From the b_{21} entry of Table 3, $b_{21} = b_{11}$ and so $\beta = b_{11}$.

$$\partial u_1/\partial a_{41} \leq \kappa(-2c_{11}(b_{11} - b_{21}) - b_{21} + b_{11})/2 = 0$$

$\Rightarrow P_1$ cannot change a_{41} from 0 to increase u_1.

Now that we have shown $a_{32} = a_{41} = 0$, we can assume $a_{32} = 0$ and complete the proof by showing that $a_{31} = 0$.

$$\begin{aligned}\partial u_1/\partial a_{31} &= \kappa[-5 + 8a_{32}(b_{21} - b_{11})c_{11} + 2a_{34}(b_{11} + b_{21}) + 3b_{11} \\ &\quad + 6b_{21} + a_{32} + 6b_{11}c_{11} - 6b_{21}c_{11} - 4a_{32}b_{21}]/2 \\ &= \kappa(-5 + 3b_{11} + 6b_{21} + 6b_{11}c_{11} - 6b_{21}c_{11})/2 \\ &\quad \text{since } a_{32} = a_{34} = 0 \\ &\leq \kappa(-5 + 3(1/4) + 6(1/4) + 6(1/4)(1/2) - 0)/2 \\ &\quad \text{by Table 3 constraint } c_{11} \leq 1/2 \text{ and by Lemma 2} \\ &= \kappa(-2)/2 < 0\end{aligned}$$

$\Rightarrow P_1$ cannot change a_{31} from 0 to increase u_1. □

PROOF. (of Theorem 2). From Theorem 1 and Lemmas 3, 4 and 5, no individual player can increase utility by unilaterally varying their strategy from the profiles listed in Table 2 and Table 3. Therefore, by the definition of Nash equilibrium, Table 2 and Table 3 define a family of Nash equilibrium profiles. □