High and higher dimensional Ellentuck spaces and initial structures in the Rudin-Keisler and Tukey types of non-p-points

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A very brief review of Tukey reduction between ultrafilters

**Def.** \( V \) is *Tukey reducible* to \( U \) (\( V \leq_T U \)) if there is a map \( f : U \to V \) such that each \( f \)-image of a filter base for \( U \) is a filter base for \( V \).

\[ U \equiv_T V \text{ iff } U \leq_T V \text{ and } V \leq_T U. \]

The Tukey equivalence class of an ultrafilter is called its *Tukey type*.

\[ V \leq_{RK} U \text{ iff } \exists f : \omega \to \omega \text{ such that } \{ f(U) : U \in U \} \text{ generates } V. \]

For ultrafilters, Rudin-Keisler reduction implies Tukey reduction. Thus, Tukey types are a coarsening of Rudin-Keisler (isomorphism) equivalence classes of ultrafilters.

For more overview, see a survey paper of Dobrinen (to appear in March in the Serbian Academy of Sciences).
Prior to work in this talk, quite a bit had been done finding embedded structures and initial structures of Tukey (and RK) types of p-points and iterated Fubini products of p-points.

[Milovich 2008 (initial work on Tukey and Isbell’s Problem)]

[Dobrinen/Todorcevic 2011 (embeddings), 2014 and 2015 (initial structures)]

[Dobrinen Continuous cofinal maps (to appear at some point in time)]

[Raghavan/Todorcevic 2012 (RK versus Tukey and first initial structure result for Ramsey ultrafilters)]

[Dobrinen/Mijares/Trujillo submitted 2014 (Boolean algebras as initial structures for Tukey and a rich amount of initial structures for RK)]

[Raghavan/Shelah submitted 2014 (embedding $\mathcal{P}(\omega)/\text{fin}$ into RK and Tukey types of p-points)]
The Forcing $\mathcal{P}(\omega \times \omega)/\text{Fin} \otimes^2$

$\text{Fin} \otimes \text{Fin} = \{ X \subseteq \omega \times \omega : \forall \infty i \in \omega \{ j \in \omega : (i, j) \in X \} \text{ is finite} \}.$

That is, for all but finitely many $i$, the $i$-th fiber of $X$ is finite.

We also use $\text{Fin} \otimes^2$ to denote $\text{Fin} \otimes \text{Fin}$.

$\mathcal{P}(\omega \times \omega)/\text{Fin} \otimes^2$ forces a generic ultrafilter $\mathcal{G}_2$ on base set $\omega \times \omega$.

$\mathcal{G}_2$ is neither a p-point, nor a Fubini product of p-points, but the projection to the first coordinates $\pi_1(\mathcal{G}_2)$ is a Ramsey ultrafilter.
In [Blass/Dobrinen/Raghavan], we showed the following:

**Thm.** In $V[G_2]$, $G_2$

(B) is a weak p-point;
(B) has the best partition property $G_2 \rightarrow (G_2)^2_{k,3}$ a non-p-point can have;
(D,R) is not Tukey maximum;
(D) $(G_2, \supseteq) \not\geq_T ([\omega_1]^{<\omega}, \subseteq)$;
(D) $G_2 >_T \pi_1(G_2)$;
(R) is not ‘basically generated’.

This left open what exactly is the structure of the Tukey types of all ultrafilters Tukey reducible to $G_2$, that is the *initial Tukey structure below $G_2$*. 
\( \mathcal{P}(\omega \times \omega) / \text{Fin}^{\otimes 2} \) is forcing equivalent to \((\text{Fin} \otimes \text{Fin})^+, \subseteq \text{Fin}^{\otimes 2}\),

which is forcing equivalent to

\( \{ X \subseteq \omega \times \omega : \text{infinitely many fibers of } X \text{ are infinite, and all finite fibers of } X \text{ are empty} \} \), partially ordered by \( \subseteq \text{Fin}^{\otimes 2} \).

We will thin this even more and put more restrictions on the subsets of \( \omega \times \omega \) we allow in order to obtain a topological Ramsey space \( \mathcal{E}_2 \) which is forcing equivalent to \( \mathcal{P}(\omega \times \omega) / \text{Fin}^{\otimes 2} \). Our space \( \mathcal{E}_2 \) looks and acts like \( \omega \) copies of the Ellentuck space, given a judiciously chosen finitization map.
Simplest Topological Ramsey Space: The Ellentuck Space

**Example.** Ellentuck space $[\omega]^\omega$. $Y \leq X$ iff $Y \subseteq X$.

Basis for topology: $[s, X] = \{Y \in [\omega]^\omega : s \sqsubset Y \subseteq X\}$.

**Def.** $\mathcal{X} \subseteq [\omega]^\omega$ is Ramsey iff for each $[s, X]$, there is $s \sqsubset Y \subseteq X$ such that either $[s, Y] \subseteq \mathcal{X}$ or $[s, Y] \cap \mathcal{X} = \emptyset$.

**Thm.** [Ellentuck 1974] Every $\mathcal{X} \subseteq [\omega]^\omega$ with the property of Baire (in the Ellentuck topology) is Ramsey.

Galvin-Prikry Theorem: All (metrically) Borel sets are Ramsey.

Silver Theorem: All (metrically) Suslin sets are Ramsey.

**Associated Forcings:** Mathias, $\mathcal{P}(\omega)/\text{fin}$.

**Associated Ultrafilter:** Ramsey ultrafilter forced by $([\omega]^\omega, \leq^*)$, has ‘complete combinatorics’.
Topological Ramsey spaces \((R, \leq, r)\)

Basic open sets: \([a, A] = \{X \in R : \exists n(r_n(X) = a) \text{ and } X \leq A\}\).

**Def.** \(X \subseteq R\) is Ramsey iff for each \(\emptyset \neq [a, A]\), there is a \(B \in [a, A]\) such that either \([a, B] \subseteq X\) or \([a, B] \cap X = \emptyset\).

**Def.** [Todorcevic] A triple \((R, \leq, r)\) is a *topological Ramsey space* if every subset of \(R\) with the Baire property is Ramsey, and if every meager subset of \(R\) is Ramsey null.

**Abstract Ellentuck Theorem.** [Todorcevic]
If \((R, \leq, r)\) satisfies A.1 - A.4 and \(R\) is closed (in \(AR^\mathbb{N}\)), then \((R, \leq, r)\) is a topological Ramsey space.

\(n\)-th Appproximations: \(AR_n = \{r_n(X) : X \in R\}\).  
Finite Approximations: \(AR = \bigcup_{n<\omega} AR_n\).
**Thm.** [DiPrisco/Mijares/Nieto (submitted 2014)] Let \( \mathcal{R} \) be a topological Ramsey space. If there exists a supercompact cardinal, then every selective coideal \( \mathcal{U} \subseteq \mathcal{R} \) is \((\mathcal{R}, \leq^*)\)-generic over \( L(\mathbb{R}) \).

The upshot is that if we show that \( \mathcal{P}(\omega \times \omega)/\text{Fin} \otimes^2 \) is forcing equivalent to some topological Ramsey space, then the above theorem implies that the generic ultrafilter \( \mathcal{G}_2 \) has ‘complete combinatorics’.
The structure behind $\mathcal{E}_2$: $(\omega^{\leq 2}, \prec)$

Let $\omega^{\leq 2}$ denote the set of non-decreasing sequences of members of $\omega$ of length less than or equal to 2.

The well-order $(\omega^{\leq 2}, \prec)$ begins as follows:

$(\emptyset) \prec (0) \prec (0,0) \prec (0,1) \prec (1) \prec (1,1) \prec (0,2) \prec (1,2) \prec (2) \prec (2,2) \prec \ldots$
Constructing the maximal member of $\mathcal{E}_2$
Figure: Maximum element $\mathbb{W}_2 \subseteq [\omega]^2$ of $\mathcal{E}_2$

Figure: $\omega^{\mathcal{X} \leq 2}$

$() \prec (0) \prec (0,0) \prec (0,1) \prec (1) \prec (1,1) \prec (0,2) \prec (1,2) \prec (2) \prec (2,2) \prec$
The space $\mathcal{E}_2$

$X \in \mathcal{E}_2$ iff $X$ is a subset of $\mathbb{W}_2$ such that
(1) $\hat{X}$ is tree-isomorphic to $\mathbb{W}_2$, and
(2) max values of the nodes of $\hat{X}$ are strictly increasing according to the wellordering $\prec$.

Note that lexicographic $\text{o.t.}(X) = \omega^2$ for each $X \in \mathcal{E}_2$.

$Y \leq X$ iff $Y \subseteq X$. 

Figure: $\mathbb{W}_2 \subseteq [\omega]^2$
Typical finite approximations to members of $\mathcal{E}_2$

Figure: $r_7(X)$

Figure: $r_{10}(Y)$
Why the funny ordering $\prec$?

It is necessary.

In order to satisfy the Amalgamation Axiom (A2 (b)) in Todorcevic’s characterization of topological Ramsey spaces, some such requirement is necessary.
(\mathcal{E}_2, \subseteq, r) \text{ is a topological Ramsey space}

**Thm.** [D] (\mathcal{E}_2, \subseteq, r) \text{ is a topological Ramsey space. Thus, every subset of } \mathcal{E}_2 \text{ with the property of Baire is Ramsey.}

**Def.** A set \( \mathcal{X} \subseteq \mathcal{E}_2 \) is Ramsey iff for each basic open \([a, X]\), there is a \( Y \in [a, X] \) such that either \([a, Y] \subseteq \mathcal{X}\) or \([a, Y] \cap \mathcal{X} = \emptyset\).

\( \mathcal{AR} \) denotes the collection of all finite approximations of members of \( \mathcal{E}_2 \).

For \( a \in \mathcal{AR} \) and \( X \in \mathcal{E}_2 \), \([a, X] := \{Y \in \mathcal{E}_2 : a \sqsubseteq Y \subseteq X\}\).

The Ellentuck topology is generated by basic open sets of the form \([a, X]\), where \( a \in \mathcal{AR} \) and \( X \in \mathcal{E}_2 \).
\( \mathcal{P}(\omega \times \omega)/\text{Fin}^{\otimes 2} \) is forcing equivalent to a new topological Ramsey space

\((\mathcal{E}_2, \subseteq \text{Fin}^{\otimes 2})\) is forcing equivalent to \(((\text{Fin}^{\otimes 2})^+, \subseteq \text{Fin}^{\otimes 2})\).

(Below any member \( A \in (\text{Fin}^{\otimes 2})^+ \) is some \( B \subseteq A \) which is an isomorphic copy of \( \mathcal{W}_2 \), and below \( B \), there is a dense subset of \((\text{Fin}^{\otimes 2})^+ \upharpoonright B \) isomorphic to \( \mathcal{E}_2 \).)
Higher order forcings

$\text{Fin} \otimes^3$ is the ideal on $\omega \times \omega \times \omega$ such that

$X \subseteq \omega^3$ is in $\text{Fin} \otimes^3$ iff for all but finitely many $i < \omega$, the $i$-th fiber of $X$ ($\{(j, k) \in \omega \times \omega : (i, j, k) \in X\}$) is in $\text{Fin} \otimes \text{Fin}$.

$\mathcal{P}(\omega^3)/\text{Fin} \otimes^3$ adds a generic ultrafilter $G_3$ on $\omega^3$ such that its projection to the first two coordinates is a generic ultrafilter forced by $\mathcal{P}(\omega^2)/\text{Fin} \otimes^2$, and its projection to the first coordinate is a Ramsey ultrafilter forced by $\mathcal{P}(\omega)/\text{Fin}$.

We thin $(\text{Fin} \otimes^3)^+$ to a topological Ramsey space $\mathcal{E}_3$ forcing equivalent (when partially ordered by $\subseteq \text{Fin} \otimes^3$) to $\mathcal{P}(\omega^3)/\text{Fin} \otimes^3$. 
The structure behind $E_3$

The well-order $(\omega^{x\leq 3}, \prec)$ begins as follows:

\[
\emptyset \prec (0) \prec (0, 0) \prec (0, 0, 0) \prec (0, 0, 1) \prec (0, 1) \prec (0, 1, 1) \prec (1) \prec (1, 1) \\
\prec (1, 1, 1) \prec (0, 0, 2) \prec (0, 1, 2) \prec (0, 2) \prec (0, 2, 2) \prec (1, 1, 2) \\
\prec (1, 2) \prec (1, 2, 2) \prec (2) \prec (2, 2) \prec (2, 2, 2) \prec (0, 0, 3) \prec \cdots
\]

(1)

Figure: $\omega^{x\leq 3}$
Figure: The maximum member of $\mathcal{E}_3$, $\mathbb{W}_3 \subseteq [\omega]^3$
The space $\mathcal{E}_3$

$X \in \mathcal{E}_3$ iff $X \subseteq \mathbb{W}_3$ and $X \cong \mathbb{W}_3$ as a tree, and also with respect to the $\prec$ order of the node labels.

$Y \preceq X$ iff $Y \subseteq X$. 

Figure: $\mathbb{W}_3$
Figure: $r_7(Y)$, a typical finite approximation to a member of $E_3$
We now define the spaces $\mathcal{E}_k$, $k \geq 2$, in general.
The well-ordered set \((\omega^{\leq k}, \prec), \ k \geq 2\).

\(\omega^{\leq k}\) denotes the collection of all non-decreasing sequences of members of \(\omega\) of length less than or equal to \(k\).

Define a well-ordering \(\prec\) on \(\omega^{\leq k}\) as follows:

() is the \(\prec\)-minimum element.

For \((j_0, \ldots, j_{p-1})\) and \((l_0, \ldots, l_{q-1})\) in \(\omega^{\leq k}\) with \(p, q \geq 1\), define 
\((j_0, \ldots, j_{p-1}) \prec (l_0, \ldots, l_{q-1})\) if and only if either

1. \(j_{p-1} < l_{q-1}\), or
2. \(j_{p-1} = l_{q-1}\) and \((j_0, \ldots, j_{p-1}) \prec_{\text{lex}} (l_0, \ldots, l_{q-1})\).

Let \(\vec{j}_m\) denote the \(\prec\)-\(m\)-th member of \(\omega^{\leq k}\).

For \(\vec{t} \in \omega^{\leq k}\), we let \(m_{\vec{t}} \in \omega\) denote the \(m\) such that \(\vec{t} = \vec{j}_{m_{\vec{t}}}\).
The spaces $\mathcal{E}_k$, $k \geq 2$

$\widehat{W}_k$ is the image of the function $\vec{I} \mapsto \{ m : \vec{j}_m \sqsubseteq \vec{I} \}, \vec{I} \in \omega^{\leq k}$.

We say that $\widehat{X}$ is an $\mathcal{E}_k$-tree if $\widehat{X}$ is a function from $\omega^{\leq k}$ into $\widehat{W}_k$ such that

(i) For each $m < \omega$, $\widehat{X}(\vec{j}_m) \in [\omega]^{\leq \vec{m}} \cap \widehat{W}_k$;
(ii) For all $1 \leq m < \omega$, $\max(\widehat{X}(\vec{j}_m)) < \max(\widehat{X}(\vec{j}_{m+1}))$;
(iii) For all $m, n < \omega$, $\widehat{X}(\vec{j}_m) \sqsubseteq \widehat{X}(\vec{j}_n)$ if and only if $\vec{j}_m \sqsubseteq \vec{j}_n$.

The space $\mathcal{E}_k$ consists of all $X := [\widehat{X}]$, where $\widehat{X}$ is an $\mathcal{E}_k$-tree.

For $X, Y \in \mathcal{E}_k$, $Y \leq X$ iff $Y \subseteq X$.

For each $n < \omega$, the $n$-th finite aproximation $r_n(X)$ is $X \cap (\{\vec{i}_p : p < n\} \times \widehat{W}_k)$, where $(\vec{i}_p : p < \omega)$ is the $\prec$-wellordering on $\omega^{\leq k}$.
The $\mathcal{E}_k$ are high dimensional Ellentuck spaces

**Thm.** [D] For each $2 \leq k < \omega$, $(\mathcal{E}_k, \subseteq, r)$ is a topological Ramsey space.

**Remarks.**

1. Each space $\mathcal{E}_{k+1}$ is comprised of $\omega$ many copies of $\mathcal{E}_k$.
2. Moreover, each projection of $\mathcal{E}_k$ to levels 1 through $j$ produces a copy of $\mathcal{E}_j$.
3. The trick was finding the right thinning and finite approximation scheme to make Axiom **A.3 (2)** hold. (The Pigeonhole Principle **A.4** was no problem.)
Initial Tukey and Rudin-Keisler structures below $\mathcal{G}_k$, $k \geq 2$

**Thm.** [D] Let $\mathcal{G}_k$ denote the generic ultrafilter forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$.

1. If $\mathcal{V} \leq_T \mathcal{G}_k$, then $\mathcal{V} \equiv_T \pi_l(\mathcal{G}_k)$ for some $l \leq k$.

2. Thus, the Tukey equivalence classes of (nonprincipal) ultrafilters Tukey reducible to $\mathcal{G}_k$ form a chain of length $k$.

3. Further, the Rudin-Keisler equivalence classes of (nonprincipal) ultrafilters RK-reducible to $\mathcal{G}_k$ form a chain of length $k$.

**Remark.** The fact that $\mathcal{E}_k$ is dense below any member of $(\text{Fin}^{\otimes k})^+$ provides a simple way of reading off the partition relations for the generic ultrafilter.
Outline of Proof

1. $(\mathcal{E}_k, \subseteq \text{Fin}^\otimes k)$ is forcing equivalent to $\mathcal{P}(\omega^k)/\text{Fin}^\otimes k$.

2. $(\mathcal{E}_k, \leq, r)$ is a topological Ramsey space.

3. Prove a Ramsey-classification theorem for equivalence relations on fronts on $\mathcal{E}_k$, extending the Pudlák-Rödl Theorem for the Ellentuck space.

4. Prove Basic Cofinal Maps Theorem, the correct analogue for our spaces of ‘every p-point having continuous Tukey reductions’.

5. For $\mathcal{V} \leq_T \mathcal{G}_k$, apply Basic Cofinal Maps Theorem to find a front $\mathcal{F}$ on $\mathcal{E}_k$ and an $f : \mathcal{F} \to \omega$ such that $\mathcal{V} = f(\langle \mathcal{G}_k | \mathcal{F} \rangle)$.

6. Apply the Ramsey-classification theorem for equivalence relations on fronts and analyze $f(\langle \mathcal{G}_k | \mathcal{F} \rangle)$. 
Def. A family of finite approximations $\mathcal{F}$ is a front on $\mathcal{E}_k$ iff

(i) $\forall X \in \mathcal{E}_k$, $\exists a \in \mathcal{F}$ such that $a \sqsubset X$; and

(ii) for $a, b \in \mathcal{F}$, $a \not\subset b$.

Def. A map $\varphi$ on a front $\mathcal{F} \subseteq \mathcal{AR}$ is called

1. inner if for each $a \in \mathcal{F}$, $\varphi(a)$ is a subtree of $\hat{a}$.

2. Nash-Williams if for all pairs $a, b \in \mathcal{F}$, whenever $\varphi(a) \sqsubseteq \varphi(b)$, then $\varphi(a) = \varphi(b)$.

3. irreducible if it is inner and Nash-Williams.
Ramsey-classification Theorem for equivalence relations on fronts

**Thm.** [D] Let $\mathcal{F}$ be a front on $\mathcal{E}_k$ and $f : \mathcal{F} \rightarrow \omega$. Then there exists an $X \in \mathcal{E}_k$ and an irreducible map $\varphi$ on $\mathcal{F}|X$ such that for all $a, b \in \mathcal{F}|X$, $f(a) = f(b)$ iff $\varphi(a) = \varphi(b)$.

**Rem.** This is the analogue (extension) of the Pudlák-Rödl Theorem for this space. Further, the canonization maps have the form that $\varphi(a)$ is a projection to some initial segments of the nodes in $a$.

**Thm.** [D] Let $R$ be an equivalence relation on some front $\mathcal{F}$ on $\mathcal{E}_k$. Suppose $\varphi$ and $\varphi'$ are irreducible maps canonizing $R$. Then there is an $A \in \mathcal{E}_k$ such that for each $a \in \mathcal{F}|A$, $\varphi(a) = \varphi'(a)$. 
Basic Cofinal Maps from $G_k$

**Def.** Given $Y \in B_k := G_k \cap E_k$, a monotone map $g : B_k|Y \to \mathcal{P}(\omega)$ is basic if there is a map $\hat{g} : \mathcal{AR}|Y \to [\omega]^{<\omega}$ such that

1. **(monotonicity)** For all $s, t \in \mathcal{AR}|Y$, $s \subseteq t \to \hat{g}(s) \subseteq \hat{g}(t)$;
2. **(initial segment preserving)** For $s \sqsubseteq t$ in $\mathcal{AR}|Y$, $\hat{g}(s) \subseteq \hat{g}(t)$;
3. **(\(\hat{g}\) represents \(g\))** For each $V \in B_k|Y$, $g(V) = \bigcup_{n<\omega} \hat{g}(r_n(V))$.

**Thm.** (Basic monotone maps on $G_k$) [D]
Let $G_k$ generic for $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$. In $V[G_k]$, for each monotone function $g : G_k \to \mathcal{P}(\omega)$, there is a $Y \in B_k$ such that $g \upharpoonright (B_k|Y)$ is basic.
**Remark.** The proofs of the Ramsey-classification Theorem for equivalence relations on fronts and the Basic Cofinal Maps Theorem could be proved using only the Abstract Nash-Williams Theorem, which we originally proved without using A.3 (2).
The sets $[\omega]^k$ are actually uniform barriers (on $\omega$) of finite rank.

Uniform barriers $B$ (on $\omega$) of any countably infinite rank provide the template for building higher order Ellentuck spaces $E_B$.

Such spaces $E_B$ are forcing equivalent to forcings constructed by continuing the process of iteratively constructing ideals built from the ideals $\text{Fin} \otimes^k$.

Rather than give all the definitions, we shall now provide an example giving the flavor of these spaces.
Let $S$ denote $\{a \in [\omega]^{<\omega} : |a| = \min(a) + 1\}$. 

$S$ is the Schreier barrier.

$\text{Fin}^S$ is an ideal on $S$: $X \subseteq S$ is in $\text{Fin}^S$ iff for all but finitely many $k < \omega$, 
$\{a \setminus \{k\} : a \in X \text{ and } \min(a) = k\} \in \text{Fin}^{\otimes k}$.

$X \subseteq S$ is in $(\text{Fin}^S)^+$ iff there are infinitely many $k$ such that 
$\{a \setminus \{k\} : a \in X \text{ and } \min(a) = k\} \in (\text{Fin}^{\otimes k})^+$.

$\mathcal{P}(S)/\text{Fin}^S$ is forcing equivalent to $((\text{Fin}^S)^+, \subseteq \text{Fin}^S)$. 

The first infinite dimensional Ellentuck space
We use the form of $\mathcal{S}$ to make our template structure of finite non-decreasing sequences of natural numbers.

$\preceq$ (0) $\preceq$ (1) $\preceq$ (1, 1) $\preceq$ (1, 2) $\preceq$ (2) $\preceq$ (2, 2) $\preceq$ (2, 2, 2) $\preceq$ (1, 3) $\preceq$ (2, 3) $\preceq$ (3, 3) $\preceq$ (3, 3, 3) $\preceq$ (3, 3, 3, 3) $\preceq$ (3, 3, 3, 3) $\preceq$ (1, 4) $\preceq$ ...
Figure: $\mathbb{W}_S$

Figure: $\omega S$
Figure: $\mathcal{W}_S$

Figure: $\omega \times S$
The space $\mathcal{E}_S$

$\mathcal{E}_S$ is the collection of all $X \subseteq \mathbb{W}_S$ such that

1. for infinitely many $k$, $\{a \setminus \{k\} : a \in X$ and $\min a = k\} \in \mathcal{E}_k$,
2. if $\{a \in X : \min a = k\} \not\in \mathcal{E}_k$, then it is empty,
3. The values of the nodes in $\hat{X}$ follow the $\prec$ order.

Figure: $\mathbb{W}_S$
In forthcoming work (still to be double checked)

**Thm.** [D]

1. The space $\mathcal{E}_S$ is a topological Ramsey space.

2. Equivalence relations on $\mathcal{AR}_1$ are canonized as projected subbarriers of $S$.

3. The Rudin-Keisler structure of all ultrafilters Rudin-Keisler reducible to the generic ultrafilter $\mathcal{G}_S$ is the ultrapower of $\mathbb{N}$ modulo the projected Ramsey ultrafilter $\pi_1(\mathcal{G}_S)$.

**Rem.** Similar theorems for all uniform barriers are obtained. Work in progress: finalizing the canonization maps for equivalence relations on fronts on $\mathcal{E}_B$ and applying them to get initial Tukey structure below $\mathcal{G}_B$, for $B$ any uniform barrier.
$\mathcal{E}_k$ ($2 \leq k < \omega$) spaces appear in


This work was initially motivated by work in


For a survey of pre-2014 results, see