I’ll give an unspecified amount of extra credit for these problems. Please talk to me about them as you work on them. I’ll date each problem so you know when the problems were added to this set. You may need to use results from problem sets or deduce preliminary results on the way to your solution.

1. **Sep 16, 2013** Here are three versions of the principle of induction. The first two were stated in class. Prove that all three versions are equivalent.

   **weak induction:** Let \( S \subseteq \mathbb{N} \) have the properties that \( 1 \in S \) and for each \( n \in \mathbb{N} \) such that \( n \in S \), then \( n + 1 \in S \). Then \( S = \mathbb{N} \).

   **strong induction:** Let \( S \subseteq \mathbb{N} \) have the properties that \( 1 \in S \) and for each \( n \in \mathbb{N} \) such that \( \{1, \cdots, n\} \in S \), then \( n + 1 \in S \). Then \( S = \mathbb{N} \).

   **well ordering of \( \mathbb{N} \):** Let \( S \subseteq \mathbb{N} \) be nonempty. Then \( S \) has a smallest element.

2. **Sep 16, 2013** If \( a > 0 \), \( b > 0 \) and \( a + b = 1 \) then

   \[
   \left( a + \frac{1}{a} \right)^2 + \left( b + \frac{1}{b} \right)^2 \geq \frac{25}{2}.
   \]

   When does equality hold?

   *(Note: This may also be proven using results from first year calculus, but, in the spirit of this course, try to do it without calculus. You may find the results of problem 6 in Problem Set 1 useful. Please include a solution of problem 6 if you use its results.)*

3. **Sep 16, 2013**

   (a) If \( a_1, a_2, \cdots, a_n > 0 \), then

   \[
   \left( \sum_{j=1}^{n} a_j \right) \left( \sum_{j=1}^{n} \frac{1}{a_j} \right) \geq n^2
   \]

   and equality holds if and only if \( a_1 = a_2 = \cdots = a_n \).

   *(Note: Part (a) can be easily proven using the Schwarz inequality. (Don’t worry if you don’t know this inequality yet.) But I suggest that you try this one by induction. My proof of the induction step uses the induction hypothesis twice!)*
(b) If \( a, b, c > 0 \) and 
\[
a + b + c = 1
\]
then 
\[
\left( \frac{1}{a} - 1 \right) \left( \frac{1}{b} - 1 \right) \left( \frac{1}{c} - 1 \right) \geq 8.
\]

4. **Sep 24, 2013** Any/all of problems 3-8 on the Cardinality Results handout qualify as advanced problems. The Schröder–Bernstein theorem is tricky. If you want to work on it, let’s have a conversation.

5. **Oct 18, 2013**

(a) Suppose that \((a_n)\) is a sequence of positive (i.e. \(a_n > 0\) for all \(n \in \mathbb{N}\)) numbers. Suppose that \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = A\). Prove that \(\lim_{n \to \infty} \sqrt[n]{a_n} = A\).

(This result is sometimes described as “the root test is better than the ratio test” where the references are to common tests for convergence of series of positive terms studied in first year calculus.)

(b) You may assume that \(\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e\), something which is true but which we have not proven. Using this, part (a) and limit rules, find
\[
\lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n}.
\]
Carefully justify your answer.

(c) Check your answer on a calculator as a reality check. For example, calculate
\[
\frac{10^{\sqrt{100!}}}{100}
\]
and see if its numerical value is close to the numerical value of your answer.

6. **Oct 22, 2013** Fix \( c < d \). Suppose that \(\{O_n : n \in \mathbb{N}\}\) is a family of nonempty open intervals and \([c, d] \subseteq \bigcup_{n=1}^{\infty} O_n\).

(a) Show that there is a finite set \(F \subseteq \mathbb{N}\) such that \([c, d] \subseteq \bigcup_{n \in F} O_n\).

(b) Recall that \(|I| = b - a\) denotes the length of the interval \(I\) with end points \(a\) and \(b\) (regardless of whether or not \(a, b \in I\)).

Show that if \(F\) is the finite set whose existence you showed in part (a), then
\[
\sum_{n \in F} |O_n| > |[c, d]|
\]
(c) Refer to problem 10, page 93. Suppose that $O$ and $F$ are the sets defined in that problem.

Prove that if $J = [c, d]$ is any closed interval and $|J| = 2$, then $F \cap J \neq \emptyset$.

Note: Here you may assume that we know that $\sum_{n=1}^{\infty} \frac{2}{2^n} = 2$. (We didn’t really study infinite series carefully.)

7. Oct 22, 2013 Suppose that $\{O_n : n \in \mathbb{N}\}$ is a family of nonempty open intervals and $(c, d) \subseteq \bigcup_{n=1}^{\infty} O_n$. (Note here the change from problem 6. The interval $(c, d)$ is now an open interval.)

(a) Show that for all $\varepsilon > 0$, there is a finite set $F \subseteq \mathbb{N}$ such that

$$\sum_{n \in F} |O_n| \geq |(c, d)| - \varepsilon$$

Hint: Can you reduce to an application of problem 6(b)?

(b) Show that $\sum_{n=1}^{\infty} |O_n| \geq |(c, d)|$. 