Here are some important results on cardinality. Some of these are in the text, some are not. Result 1 is shown in class, result 2 is in problem set 3, and the others are on the Advanced Problems sheet. As always, we’re happy to consult about these.

**Here are the results:**

1. A subset $A$ of a countable set is finite or countable.

2. Suppose that $B$ is an infinite set and $f: \mathbb{N} \to B$ is an onto function. Prove that $B$ is countable.

3. Suppose that $B$ is a countable subset of $\mathbb{R}$. Then $B^c = \{ x \in \mathbb{R} : x \notin B \}$ is uncountable. Conclude that the set of irrational numbers is uncountable. *(You might start by showing that the union of two countable sets is a countable set.)*

4. A countable union of finite or countable sets is finite or countable. Specifically, if $A_n$ is a finite or countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is finite or countable.

   *Give an example where each $A_n$ is finite and $\bigcup_{n=1}^{\infty} A_n$ is also finite to see that this situation can occur. Give an example where each $A_n$ is finite and $\bigcup_{n=1}^{\infty} A_n$ is countable to see that this situation can occur.*

5. If $A$ and $B$ are finite sets, then $A \times B$ is a finite set.

6. If $A$ and $B$ are countable sets, then $A \times B$ is a countable set.

7. For a set $A$, $\mathcal{P}(A)$ denotes the set of all subsets of $A$ and $\mathcal{P}(A)$ is called the *power set* of $A$. Prove that if $A \neq \emptyset$, then there is no onto map $f: A \to \mathcal{P}(A)$.

   *To get intuition about this concept, check that $\mathcal{P}(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.*

8. ★ **Schröder–Bernstein Theorem:** Suppose that $A$ and $B$ are nonempty sets. Suppose also that

   - there is a one-to-one function $f: A \to B$; and
   
   - there is a one-to-one function $g: B \to A$.

   Then there is a bijection $h: A \to B$. I.e. $|A| = |B|$.

   *(Comment: The proof requires a definition of the function $h$ and then a proof that $h$ is both one-to-one and onto.)*