1. **Banach Algebras and Spectral Theory**

1.1. **Gelfand–Mazur Theorem.** We denote this section in the above manner since this result is the culmination of many basic results in spectral theory.

**Theorem 1.1.0.1** (Gelfand–Mazur Theorem). *Every division ring unital Banach Algebra is \( \mathbb{C} \).*

Recall that a division ring is a ring in which every non-zero element is invertible, and the “is” in the statement refers to an isometric isomorphism. Note that for our purposes, we will prove by the end of
this section an equivalent theorem. It is a very simple exercise to show that the above theorem and the following are equivalent. We will show that "every simple abelian unital Banach Algebra is $\mathbb{C}$." For notational purposes, we will denote the unit of our Banach Algebra by $e$ if the context is clear.

**Notation 1.1.0.1.** Let $\text{Inv}(\mathcal{B})$ denote the invertible elements of a Banach Algebra, $\mathcal{B}$.

**Lemma 1.1.0.1.** If $\mathcal{B}$ is a Banach Algebra and $b \in \mathcal{B}$ such that $\|b\| < 1$, then

i. $e - b \in \text{Inv}(\mathcal{B})$ and $(e - b)^{-1} = \sum_{n=0}^{\infty} b^n$,

ii. $\text{Inv}(\mathcal{B})$ is open and $x \mapsto x^{-1}$ is continuous.

**Proof**. **proof of i**: Let $b$ be as above, then by geometric series, $\sum_{n=0}^{\infty} b^n < \infty$. But, since we are in a Banach Space, this implies that $\sum_{n=0}^{\infty} b^n = a$, where $a < \infty$. Let $a_N$ denote the $N^{th}$-partial sum. Now,

$$(e - b)a_N = e \cdot a_N - b \cdot a_N$$

$$= a_N \cdot e - b \cdot \sum_{n=0}^{N} b^n$$

$$= \sum_{n=0}^{N} b_n - \sum_{n=1}^{N+1} b^n$$

$$= e - b^{N+1}$$

But, since $b$ commutes with powers of itself, $(e - b)a_N = a_N(e - b) = e - b^{N+1}$. Next, taking limits, we have since by divergence test $b^n \to 0$, $(e - b)a = a(e - b) = e$. Thus, $e - b$ is invertible and its inverse $(e - b)^{-1} = a$.

**proof of ii**: We will start by showing the inverse is continuous and in the process obtain the necessary approximations to show that $\text{Inv}(\mathcal{B})$ is open.

So, let $b \in \text{Inv}(\mathcal{B})$ and $\varepsilon > 0$. Thus, $b^{-1} \in \text{Inv}(\mathcal{B})$ exists and its norm is nonzero. Choose

$$\delta = \frac{1}{2} \min\{\|b^{-1}\|^{-1}, \varepsilon\|b^{-1}\|^{-1}\}$$

Let $a \in B_{\delta}(b)$. Notice that

$$b(e - b^{-1}(b - a)) = b \cdot e - b \cdot b^{-1}(b - a)$$

$$= b - e(b - a) = b - b + a$$

$$= a$$
Now,
\[ \|b^{-1}(b-a)\| \leq \|b^{-1}\| \cdot \|b-a\| = \|a^{-1}\| \delta \leq \frac{1}{2} < 1 \]
Hence, by part i., \( e - b^{-1}(b-a) \) is invertible. Thus, since the invertible elements form a group under multiplication and \( b \) is invertible, \( a = b(e - b^{-1}(b-a)) \) is invertible. Now, notice that for this step our \( \delta \) did not need to depend on \( \epsilon \). We only required that \( \delta = \frac{1}{2}||b^{-1}||^{-1} > 0 \). Therefore, we have shown that for any invertible \( b \), there exists a ball of radius \( \delta > 0 \) contained in \( \text{Inv}(B) \). So, the set is open.

Now, back to continuity, \( a^{-1} = (e - b^{-1}(b-a))^{-1}b^{-1} \). But, again by part i., geometric series, and the above inequality,
\[
\|(e - b^{-1}(b-a))^{-1}\| = \|\sum_{n=0}^{\infty} (b^{-1}(b-a))^{n}\|
\leq \sum_{n=0}^{\infty} \|(b^{-1}(b-a))^{n}\|
= \frac{1}{1 - \|(b^{-1}(b-a))\|}
< 2
\]
Thus,
\[ \|a^{-1}\| = \|(e - b^{-1}(b-a))^{-1}b^{-1}\| < 2\|b^{-1}\|. \]
Hence, we found a \( \delta > 0 \) dependent on \( \epsilon > 0 \) and \( b \) such that for all \( a \in B_{\delta}(b) \),
\[
\|a^{-1} - b^{-1}\| = \|a^{-1}(b-a)b^{-1}\|
\leq \|b^{-1}\| \cdot \|a^{-1}\| \cdot \|b-a\|
\leq 2\|b^{-1}\|^{2}\delta
\leq \epsilon
\]
\[ \square \]

Now, the next result is the result for which our version of Gelfan’d - Mazur will be presented as a Corollary to. But, first, a definition.

**Definition 1.1.0.1.** The spectrum of an element \( x \) in a unital Banach Algebra, \( B \), is the set \( \sigma(x) = \{ \lambda \in \mathbb{C} : \lambda - x \notin \text{Inv}(B) \} \), where \( \lambda \) first denotes both the complex number and then the scalar multiple of the identity, \( \lambda \cdot e \).

**Proposition 1.1.0.1.** If \( B \) is a unital Banach Algebra and \( x \in B \), then \( \sigma(x) \) is compact and non-empty.
Proof. Fix \( x \in \mathcal{B} \). First, compactness. We need to show that \( \sigma(x) \) is closed and bounded. Assume that \( \sigma(x) \neq \emptyset \), we would like to show that \( \sigma(x) \) is bounded by circle about the origin of radius \( ||x|| \). Thus, assume by way of contradiction that there exists \( \lambda \in \sigma(x) \) such that \( ||x|| < |\lambda| \). Then, \( ||x/\lambda|| < 1 \). Thus by the first part of Lemma 1.1.0.1, \( e - x/\lambda \in \text{Inv}(\mathcal{B}) \). But, since \( \lambda \in \text{Inv}(\mathcal{B}) \) and it is a group, \( \lambda - x = \lambda(e - x/\lambda) \in \text{Inv}(\mathcal{B}) \), a contradiction. Thus, \( \sigma(x) \) is bounded.

Next for closed, define the map \( R : \mathbb{C} \longrightarrow \mathcal{B} \) by \( R(\lambda) = \lambda - x \), which is \( 1 - \text{Lipschitz} \) and therefore continuous. Also, \( R^{-1}(\text{Inv}(\mathcal{B})) = \{ \lambda \in \mathbb{C} : R(\lambda) \in \text{Inv}(\mathcal{B}) \} \)

\[ = \{ \lambda \in \mathbb{C} : \lambda - x \in \text{Inv}(\mathcal{B}) \} \]

\[ = \mathbb{C} \setminus \sigma(x) \]

Thus, by part ii. of Lemma 1.1.0.1, \( \sigma(x) \) is the complement of an open set an therefore is closed.

Now, for non-empty, assume by way of contradiction that \( \sigma(x) = \emptyset \). To reach this contradiction, we use Liouville’s Theorem. SO, we construct a complex function based on our \( x \) with empty spectrum that satisfies the hypotheses of Liouville. Thus, \( \mathbb{C} = \mathbb{C} \setminus \sigma(x) \). Hence, \((\lambda - x)^{-1}\) is defined for all \( \lambda \in \mathbb{C} \) by definition of spectrum. So, we defined the map \( g : \mathbb{C} \longrightarrow \mathcal{B} \) by \( g(\lambda) = (\lambda - x)^{-1} \). Again, by part ii. of the previous lemma, \( g \) is continuous by composition of the subtraction and the inverse map which are both continuous. Thus, notice that if \( \lambda \to \lambda_0 \), then \( g(\lambda) \to g(\lambda_0) \) or \((x - \lambda)^{-1} \to (x - \lambda_0)^{-1} \). Therefore,

\[ \lim_{\lambda \to \lambda_0} \frac{g(\lambda) - g(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \to \lambda_0} \frac{(x - \lambda)^{-1} - (x - \lambda_0)^{-1}}{\lambda - \lambda_0} \]

\[ = \lim_{\lambda \to \lambda_0} \frac{((x - \lambda_0) - (x - \lambda))(x - \lambda_0)^{-1}}{(x - \lambda_0)^{-1}} \]

\[ = \lim_{\lambda \to \lambda_0} (x - \lambda_0)^{-1} \]

\[ = (x - \lambda_0)^{-2} \]

\[ \square \]

Now, since \( \mathcal{B} \) is a Banach Space, let \( \rho \) be in its dual \( \mathcal{B}' \). Define \( f : \mathbb{C} \longrightarrow \mathbb{C} \) by \( f(\lambda) = \rho(g(\lambda)) \). By the previous calculation, the linearity and continuity of \( \rho \), and that \( g \) was defined everywhere by assumption, \( f \) is differentiable everywhere and therefore entire. And, its derivative and some \( \lambda \in \mathbb{C} \) is \( f(\lambda) = \rho((x - \lambda)^{-2}) \).
The last hypothesis for Liouville is that \( f \) must be bounded. Note that by \( \rho \in B' \),
\[
|f(\lambda)| = |\rho(g(\lambda))| \leq k\|g(\lambda)\| = k\|(x - \lambda)^{-1}\|.
\]
Now, we will be considering \( \lambda \in \mathbb{C} \) such that \( |\lambda| \to \infty \). Thus, choose \( \lambda \in \mathbb{C} \) such that \( |\lambda| > ||x|| \) since our \( x \) has been fixed since the beginning of the argument, then \( ||x/\lambda|| < 1 \). Thus, by part i. of the previous lemma,
\[
||(x - \lambda)^{-1}|| = \frac{1}{|\lambda|}||(e - x/\lambda)^{-1}|| = \frac{1}{|\lambda|} \sum_{n=0}^{\infty} (x/\lambda)^n \leq \frac{1}{|\lambda|} \sum_{n=0}^{\infty} ||x/\lambda||^n = \frac{1}{|\lambda|} = \frac{1}{|\lambda| - ||x||} = \frac{1}{|\lambda| - ||x||}
\]
So, for such \( \lambda \), \( |f(\lambda)| \leq k \cdot \frac{1}{|\lambda| - ||x||} \). Therefore, \( |f(\lambda)| \to 0 \) as \( |\lambda| \to \infty \). Hence, since \( f \) is continuous by differentiability and vanishes at infinity, it is bounded. Thus, by Liouville, \( f \) is constant and by vanishing this constant must be \( 0 \). Thus, for all \( \rho \in B' \), \( \rho(g(\lambda)) = 0 \). Thus, by Hahn-Banach, \( g(\lambda) = 0 \) but this implies that \( (x - \lambda)^{-1} = 0 \), but \( 0 \) is not invertible, which is a contradiction. Thus, \( \sigma(x) \) is nonempty.

**Corollary 1.1.0.1.** Every simple abelian unital Banach Algebra is \( \mathbb{C} \).

**Proof.** Let \( B \) be such a Banach Algebra. If we show that every element in \( B \) is of the form \( \lambda e \) for some \( \lambda \in \mathbb{C} \), then the obvious map would be an isometric isomorphism to \( \mathbb{C} \). Thus, assume by way of contradiction that there is some \( a \in \mathbb{C} \) such that \( a \neq \lambda e \) for all \( \lambda \in \mathbb{C} \). In other words, \( a - \lambda \neq 0 \) for all \( \lambda \in \mathbb{C} \). By the previous proposition and the previous comment, there exists some \( \alpha \in \sigma(x) \) such that \( x - \alpha \neq 0 \) and \( x - \alpha \not\in \text{Inv}(B) \). Thus, \( X = \overline{(x - \alpha)B} \) is a non-zero ideal. Next, we claim that no element in \( X \) is invertible. First, we show that no element in \( (x - \alpha)B \) is invertible. Assume by way of contradiction that there exists \( 0 \neq b \in B \) such that \( (x - \alpha)b \in \text{Inv}(B) \). So, there exists some \( c \in B \) such that \( (x - \alpha)bc = e \) and \( c(x - \alpha)b = e \). But, since \( B \) is abelian this would imply that \( bc \) is a left and right inverse for \( x - \alpha \), which is a contradiction.
Now, for all $b \in B$, $(x - \alpha)b \notin \text{Inv}(B)$ implies that $e - (e - (x - \alpha)b) \notin \text{Inv}(B)$. Thus, by the contrapositive of part i. of Lemma 1.1.0.1,

$$
\|e - (x - \alpha)b\| \geq 1.
$$

Hence, no net from $(x - \alpha)B$ could converge to $e$ in norm and so $e \notin X$, which would imply that $X$ is a non-zero proper ideal, which is a contradiction to simple. Therefore, every element in $B$ is a scalar multiple of $e$, and we are finished.

Notice that this is also true for $C^*$-algebras. One must only have to check that the obvious map is an isometric *-isomorphism but this is trivial since $(\lambda e)^* = \overline{\lambda} e$. For the sake of completion, we finish with a proof of the Gelfan’d - Mazur Theorem in its more common phrasing, which more directly uses the fact the spectrum is non-empty. The above phrasing is due to page 5 of Davidson’s text [2]. He utilizes this phrasing since results about the Maximal Ideal space will use the previous theorem directly, which is more useful than the common phrasing.

**Theorem 1.1.0.2** (Gelfan’d - Mazur Theorem). Every division ring unital Banach Algebra is $\mathbb{C}$.

**Proof.** Assume by way of contradiction that there is some $b \in \mathbb{C}$ such that $b - \lambda \neq 0$ for all $\lambda \in \mathbb{C}$. But, by Proposition 1.1.0.1, its spectrum is non-empty. So, there exists some $\beta \in \mathbb{C}$ such that $x - \beta \notin \text{Inv}(B)$. Yet, $x - \beta \neq 0$, which would imply the existence of a non-zero non-invertible element, which would contradict division ring. So, by the same reasoning as in the previous proof, the obvious map would be an isometric isomorphism.

Notice that we did not need abelian in the previous argument, but we needed abelian for our first phrasing to construct an ideal, which had no invertible elements.

**2. Commutative $C^*$-Algebras**

The main goal of this section is to develop the Continuous Functional Calculus for normal elements in $C^*$-algebras. That is, to prove that for any normal element $N$ in a $C^*$-algebra, the $C^*$-algebra generated by $N$ is isometrically *-isomorphic to the continuous functions on its spectrum in the unitilization which vanish at $0 \in \mathbb{R}$. 
The first lemma is a weak version of the Spectral Mapping Theorem, which will be a result provided by the Continuous Functional Calculus. This much stronger result will provide equality and allow for arbitrary continuous functions in the following lemma. Note that when we apply a polynomial from a $C^*$-algebra to itself to the spectrum, we are viewing $p(\mu)$ as $p(\mu e)$ where $e$ is the unit.

**Lemma 2.1.0.2.** If $A$ is a unital $C^*$ algebra, then for any $a \in A$ and $p$ polynomial, $p(\sigma(a)) \subseteq \sigma(p(a))$.

**Proof.** Let $\mu \in \sigma(a)$. We show that $p(\mu) \in \sigma(p(a))$. Now, the function $p(z) - p(\mu)$ is a polynomial with a root at $z = \mu$. Thus, $(z - \mu)$ divides $p(z) - p(\mu)$. Let $q$ be the quotient so that $p(z) - p(\mu) = q(z)(z - \mu) = (z - \mu)q(z)$. Thus, $p(a) - p(\mu) = q(a)(a - \mu) = (a - \mu)q(a)$. Now, if $p(a) - p(\mu)$ were invertible with inverse $b$, then by the above equality, $a - \mu$ would have a left and right inverse, a contradiction. Thus, $p(a) - p(\mu)$ is not invertible, and so $p(\mu) \in \sigma(p(a))$. \qed

The next proposition will become useful when showing that we have an isometry to the continuous functions mentioned above for a commutative $C^*$-algebra. First, we define a new expression.

**Definition 2.1.0.2.** The **spectral radius** is defined as $r(a) = \sup\{|\mu| : \mu \in \sigma(a)\}$.

Notice that in the proof that the spectrum is compact, we showed that it was bounded and in fact $r(a) \leq \|a\|$.

**Proposition 2.1.0.2.** $r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$.

**Proof.** Let $\mu \in \sigma(a)$. Then since $x^n$ is a polynomial, by the previous result, $\mu^n \in \sigma(a^n)$. Hence, by the inequality introduced immediately before this proposition and by definition of the spectral radius $r(a^n)$,

$$|\mu|^n = |\mu^n| \leq r(a^n) \leq \|a^n\|.$$  

By taking $n$th roots since $\mu \in \sigma(a)$ was arbitrary $r(a) \leq \liminf_{n} \|a^n\|^{1/n}$.

It remains to show that $\limsup_{n} \|a^n\|^{1/n} \leq r(a)$. Choose $\mu \in \mathbb{C}$ such that $|\mu| < 1/r(a)$. If $r(a) = 0$ then we may choose $\mu$ arbitrarily. Now, if we show that $\|a^n\|^{1/n} \leq 1/|\mu|$ for all $n$ and fixed $\mu$, we would be finished since $\limsup_{n} \|a^n\|^{1/n} \leq 1/|\mu|$ and next we make take an increasing limit.
as $|\mu| \to 1/r(a)$ to bound $\|a^n\|^{1/n}$ above by $r(a)$. Notice that

\[
\|a^n\|^{1/n} \leq 1/|\mu| \iff \|a^n\| \leq 1/|\mu|^n
\]

\[
\iff |\mu|^n \cdot \|a^n\| \leq 1
\]

\[
\iff \|\mu^n a^n\| \leq 1
\]

\[
\iff \|(\mu a)^n\| \leq 1
\]

Thus, if we show that $\{((\mu a)^n)\}_{n=1}^\infty$ is bounded in norm by 1, then we would be done. But, even better, since for any $M > 0$, $M^{1/n} \to 1$, we need to only show that $\{((\mu a)^n)\}_{n=1}^\infty$ is bounded in norm. Since going through the above equivalence backwards we would have that $\|a^n\|^{1/n} \leq M^{1/n}/|\mu|$ for all $n$, which would still bound $\|a^n\|^{1/n}$ above by $1/|\mu|$ and therefore by $r(a)$. To show the sequence $\{((\mu a)^n)\}_{n=1}^\infty$ is bounded in norm, we use the following formulation of Banach-Steinhaus: For a normed space $\mathcal{X}$ and $A \subseteq \mathcal{X}$, $A$ is bounded in norm if and only if for every $\rho \in \mathcal{B}'$, $\sup\{|\rho(a)| : a \in A\} < \infty$. This may be found on page 96 in [1]. So, we take $A = \{((\mu a)^n)\}_{n=1}^\infty$.

Let $\rho \in \mathcal{B}'$. We show $\{\rho(a^n)\mu^n\}_{n=1}^\infty$ is a bounded sequence. Let $z \in \mathbb{C}$ such that $|z| < 1/|\mu|$, then $\|z\mu\| < 1$. Thus by Lemma 1.1.0.1 $(e - za)^{-1}$ exists and is equal to $\sum_{n=0}^\infty (za)^n = \sum_{n=0}^\infty z^n a^n$ since $z \in \mathbb{C}$. Now, by continuity and linearity of $\rho$, $\rho((e - za)^{-1}) = \sum_{n=0}^\infty \rho(z)^n a^n < \infty$. Thus, $\{|\rho(z)^n a^n\| \}_{n=1}^\infty$ is a bounded sequence for $|z| < 1/|\mu|$, which is an open set contained in the open set $M = \{z : |z| < 1/r(a)\}$ since $r(a) \leq |\mu|$. Now, if we show the above series converges for $z \in M$ then we would be able to take $z = \mu$ and be finished.

Notice though that we just showed that $f(z) = \rho((e - za)^{-1})$ is analytic on the open set for which $|z| < 1/|\mu|$ with the above power series expansion centered at 0. Thus, if we show that $f$ is analytic on $M$, then $f$ must have the same power series expansion for $z \in M$, which would finish the proof. Assume that $z \in M$ for which $z \neq 0$. Thus, $r(a) < 1/|z|$ and so it couldn’t be the case that $z^{-1} \in \sigma(a)$. With only this assumption in the proof of $\sigma(a) \neq \emptyset$, we showed that $\rho((z^{-1} - a)^{-1}$ is differentiable and therefore analytic. But, $f(z) = z^{-1}\rho((z^{-1} - a)^{-1}$, which as a scalar multiple of a differentiable function is differentiable, which is again analytic. Hence, by Banach-Steinhaus, $A = \{((\mu a)^n)\}_{n=1}^\infty$ is bounded in norm, and by the previous comments, we are finished since $\mu$ was chosen arbitrarily such that $|\mu| < 1/r(a)$.

Now, onto maximal ideals.

**Definition 2.1.0.3.** The **Maximal Ideal Space** of a commutative $C^*$-algebra $\mathcal{A}$ is

\[
\Psi = \{\psi \in \text{Hom}(\mathcal{A}, \mathbb{C}) : \psi \neq 0\}.
\]
This space will be the space for which \( C(\Psi) \) will be isometrically *-isomorphic to the unital commutative \( C^* \)-algebra for which \( \Psi \) is defined for. But, for \( C(\Psi) \) to make sense we must show that it is compact Hausdorff in the unital case.

**Proposition 2.1.0.3.** If \( A \) is a unital commutative \( C^* \)-algebra, then \( \Psi \) is compact Hausdorff in the weak* topology.

**Proof.** We show that \( \Psi = \{ \psi \in \text{Hom}(A, \mathbb{C}) : \| \psi \| = 1 = \psi(e) \} := S \). Thus, since each property is closed under converging nets including multiplicative, we would be finished. Clearly, \( S \subseteq \Psi \). Let \( \psi \in \Psi \). Assume that \( \psi(e) = \nu \). Now, \( \psi(a) \neq 0 \) for some \( a \in A \). Thus, \( \psi(a) = \psi(ae) = \psi(a)\psi(e) = \psi(a)\nu \). Hence, \( \psi(a)(\nu - 1) = 0 \). By \( \psi(a) \neq 0 \), \( \nu - 1 = 0 \implies \nu = 1 \). Hence, \( \| \psi \| \geq 1 \).

Now, assume by way of contradiction that \( \| \psi \| > 1 \). By definition of dual norm, there exist \( a \in A \) such that \( \| a \| < \psi(a) \). If \( \psi(a) \geq 1 \), then choose \( k \in \mathbb{R} \), \( k > 0 \) such that \( k\psi(a) = 1 \). Thus, \( \| ka \| \leq \psi(ka) = 1 \). By the proof of **Lemma 1.1.0.1** there exists \( b \in A \) such that \( (e - ka)b = e \). Thus, \( b = e + kab \). Hence, \( \psi(b) = \psi(e + kab) = \psi(e) + \psi(ka)\psi(b) = 1 + \psi(b) \implies 1 = 0 \), a contradiction. Thus, \( \| \psi \| = 1 = \psi(e) \). Therefore, \( S = \Psi \).

\( \Psi \) is a weak*-closed subset of the unit ball in \( A' \), so by Banach-Alaoglu, \( \Psi \) is closed in a compact set and is therefore compact. Hausdorff is inherited by the weak* topology.

\( \square \)

Therefore, for a unital abelian \( C^* \)-algebra, \( C(\Psi) \) makes sense.

The following lemma will allow us to compare the spectrum of an element to \( \Psi \) and will also be useful once we drop the unit. Also, it give meaning to the naming of \( \Psi \).

**Lemma 2.1.0.3.** There exists a bijection between \( \Psi \) and the maximal ideals for a unital \( C^* \)-algebra.

**Proof.** We show that the map \( \psi \mapsto \ker \psi \) is a bijection. Since \( \psi \) is a complex homomorphism, when we quotient \( A \) with \( \ker \psi \), we get the complex numbers, a field. Thus, \( \ker \psi \) is Maximal. So, the map is well-defined.

Next assume that \( \ker \psi = \ker \phi = M \), for \( \psi, \phi \in \Psi \). We show that \( \psi, \phi \) agree on \( A \setminus M \). Assume that \( a \notin M \). Then, \( \psi(a), \phi(a) 
eq 0 \). Let \( \psi(a) = k \), then since \( \psi(e) = 1 \), \( \psi(a - ke) = 0 \). So, \( a - \psi(a)e = a - ke \in M \).
Thus, the map is injective.

Now, let $M$ be a maximal ideal. Now, since $e \notin M$, by a similar argument to the proof of the simple version of the Gelfand-Mazur theorem, $\text{dist}(e, M) \geq 1$. Thus, the closure of $M$ cannot contain $e$. But, the ideal properties of $M$ are still preserved in its closure, so that the closure of $M$ is still a proper ideal. Thus, by maximality of $M$, $M$ equals its closure. Thus, it is a closed ideal. But, this implies that $A/M$ is a unital abelian Banach Algebra, which is simple by maximality. Thus, by the simple Gelfand-Mazur Theorem, $A/M$ is $\mathbb{C}$. Thus, the quotient map $\psi : A \to A/M = \mathbb{C}$ is a non-zero complex homomorphism, and furthermore, $\ker \psi = M$. The map in question is onto and therefore a bijection. □

Note that it takes a lot more work to show that $A/M$ is a $C^\ast$-algebra (that every $C^\ast$-algebra contains an approximate identity), but we did not need this fact here. This will be covered in a later section before we cover representations since we need such a fact for representations and for their kernels and quotients to work well with $C^\ast$-algebra theory, i.e., a First Isomorphism Theorem for $C^\ast$-algebras.

Now, that we have a better handle on $\Psi$, we make the following definition.

**Definition 2.1.0.4.** The **Gelfand Map** on a unital commutative $C^\ast$-algebra is defined as

$$\Gamma : A \to C(\Psi),$$

where $\Gamma(a) = \hat{a}$. $\hat{a}$ is the evaluation map and is continuous by definition of the weak* topology. Hence, $\Gamma$ is well-defined.

Also, it is clear that $\Gamma$ is a homomorphism since each $\psi \in \Psi$ is a homomorphism. Another note is that $\Gamma(e) = \hat{e}$, but $\hat{e}(\psi) = \psi(e) = 1$. Thus, $\Gamma(A)$ is a unital subalgebra of $C(\Psi)$.

But, before we move on to prove that $\Gamma$ is an isometric $\ast$-isomorphism, we need more machinery, in which we compare $\Psi$ with the spectrum. $\Psi$ is sometimes called the spectrum of a $C^\ast$-algebra. The following lemma provides an explanation.

**Lemma 2.1.0.4.** $\sigma(a) = \sigma(\hat{a}) = \{\psi(a) : \psi \in \Psi\}$, and so $\|\hat{a}\|_\infty = r(a)$. 

Proof. To show that for $a \in A$, a unital commutative $C^*$-algebra, $\sigma(a) = \sigma(\hat{a})$, we show something a little stronger. We show that $a \in \text{Inv}(A) \iff \hat{a} \in \text{Inv}(\mathcal{C}(\Psi))$.

For the forward direction, assume that $a$ is invertible. Then since $\Gamma$ is a homomorphism $\hat{a}^{-1} = \Gamma(a)^{-1} = \Gamma(a^{-1})$. Thus, $\hat{a}$ is invertible.

For the other direction, assume that $a$ is not invertible. By a similar argument to the proof of the Simple Gelfand-Mazur theorem, $I = aA$ does not contain $e$, and is therefore a two-sided proper ideal. By a Zorn argument (this is true for all proper ideals not just left or right generated), $I$ is contained in a maximal ideal $M$. The previous result implies that there is a unique $\psi \in \Psi$ sucht that $\ker \psi = M$. Since $a \in M$, $\hat{a}(\psi) = \psi(a) = 0$, and $\hat{a}$ is not invertible in $\mathcal{C}(\Psi)$. Thus, for $\mu \in \mathbb{C}$.

$$a - \mu \notin \text{Inv}(A) \iff \hat{a} - \mu \notin \text{Inv}(\mathcal{C}(\Psi))$$

So that, $\sigma(a) = \sigma(\hat{a})$.

It remains to show that $\sigma(\hat{a}) = \{\psi(a) : \psi \in \Psi\}$. Let $\mu \sigma(\hat{a})$, then $\hat{a} - \mu$ is not invertible. Thus, there exists some $\phi \in \Psi$ such that $\frac{\hat{a} - \mu(\phi)}{\phi} = 0 \implies \phi(a - \mu) = 0 \implies \phi(a) - \mu = 0 \implies \phi(a) = \mu$. Hence, $\mu \in \{\psi(a) : \psi \in \Psi\}$.

For the reverse inclusion, let $\phi(a) \in \{\psi(a) : \psi \in \Psi\}$. Now,

$$(\hat{a} - \phi(a)) \phi = a - \phi(a) \phi$$

$$= \phi(a - \phi(a))$$

$$= \phi(a - \phi(a) e)$$

$$= \phi(a) - \phi(a) \phi(e)$$

$$= \phi(a) - \phi(a) = 0$$

Thus, the map $\hat{a} - \mu$ is not invertible, so that $\mu \in \sigma(\hat{a})$. In summation, we have shown that $\sigma(a) = \sigma(\hat{a}) = \{\psi(a) : \psi \in \Psi\}$.

Lastly,

$$\|\hat{a}\|_{\infty} = \sup\{|\hat{a}(\psi)| : \psi \in \Psi\}$$

$$= \sup\{|\mu| : \mu \in \sigma(\hat{a})\}$$

$$= \sup\{|\mu| : \mu \in \sigma(a)\}$$

$$= r(a)$$

\[\square\]
2.2. Gelfand-Naimark for Commutative $\mathcal{C}^*$-algebras.

**Theorem 2.2.0.3.** If $\mathcal{A}$ is a unital commutative $\mathcal{C}^*$-algebra, then $\mathcal{A}$ is isometrically $\ast$-isomorphic to $\mathcal{C}(\Psi)$.

**Proof.** We will show that $\Gamma : \mathcal{A} \rightarrow \mathcal{C}(\Psi)$ is an isometric $\ast$-isomorphism. We have already established that it is a well-defined homomorphism.

We will next show that it is onto, but in the process we will accidentally show all the other requirements for such a map. To show that it is onto, we show that $\Gamma(\mathcal{A}) = \mathcal{C}(\Psi)$ by showing that $\Gamma(\mathcal{A})$ is dense and closed in $\mathcal{C}(\Psi)$. To show that it is dense, we use Stone-Weierstrass. Hence, we must show that $\Gamma(\mathcal{A})$ is a unital self-adjoint subalgebra which separates points. Self-adjoint subalgebra will come from showing that $\Gamma$ is in fact a $\ast$-homomorphism. To show that $\Gamma(\mathcal{A})$ is closed, we show that $\Gamma$ is an isometry, but this also provides that the map is injective. Thus, we will have an onto isometric $\ast$-homomorphism, which is all we need.

We start with density. By the remarks after the definition of $\Gamma$, $\Gamma(\mathcal{A})$ is a unital subalgebra of $\mathcal{C}(\Psi)$. For separates points, take $\psi \neq \phi \in \Psi$. $\phi(a) \neq \psi(a)$ for some $a \in \mathcal{A}$. Hence, $\Gamma(a)(\psi) = \hat{a}(\psi) \neq \hat{a}(\phi) = \Gamma(a)(\phi)$.

Next we show self-adjoint, so we verify that $\Gamma(a^\ast) = \Gamma(a)^\ast$. But, by the involution in $\mathcal{C}(\Psi)$, point-wise conjugation,

$$\Gamma(a^\ast) = \Gamma(a)^\ast \iff \forall \psi \in \Psi, \hat{a}^\ast(\psi) = \overline{\hat{a}(\psi)}$$

$$\iff \forall \psi \in \Psi, \psi(a^\ast) = \overline{\psi(a)}$$

Fix $\psi \in \Psi$, we first assume that $a \in \text{sa}(\mathcal{A})$. In this case, $\psi(a) = \psi(a^\ast)$, so we show that $\psi(a) = \overline{\psi(a)}$, which is to prove that $\psi(a) \in \mathbb{R}$. Let $\psi(a) = r + it$, where $r, t \in \mathbb{R}$. If $e^{-st} \leq 1$ for all $s \in \mathbb{R}$, then $t = 0$. Note that $e^b$ for $b \in \mathcal{A}$ as a powers series is well-defined since in a Banach Space absolute convergence implies convergence, and it is easy to show that $e^b$ converges absolutely. So, we may define for $s \in \mathbb{R}$,

$$U_s = e^{isa} = \sum_{n=0}^{\infty} \frac{(isa)^n}{n!}.$$  

Next, by the fact that $\ast$ is an additive isometry with $i^\ast = -i$, which is multiplicative on self-ajoint elements, and recall $a = a^\ast$,

$$U_s^\ast = \sum_{n=0}^{\infty} \frac{(-isa)^n}{n!} = e^{-isa}.$$  

Hence, since $isa$ and $-isa$ commute, we may add exponents,

$$U_s U_s^\ast = e^{isa} e^{-isa} = e^{isa-isa} = e^0 = 1.$$
So, that $U_s$ is unitary by commutativity, and by the $C^*$-identity, $\|U_s\| = 1$. Notice further that by the continuity, linearity, and multiplicativity of $\psi$, $\psi(U_s) = \psi\left(\sum_{n=0}^{\infty} \frac{(isa)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(i\psi(a))^n}{n!} = e^{i\psi(a)}$. Therefore, since $\|\psi\| = 1$ and the definition of the dual norm,

$$1 \geq |\psi(U_s)|$$

$$= |e^{i\psi(a)}|$$

$$= |e^{i(s(r+it))}|$$

$$= |e^{-st+i\rho r}|$$

$$= |e^{-st}e^{i\rho r}|$$

$$= e^{-st}$$

By previous comments, since $s \in \mathbb{R}$ was arbitrary, $t = 0$, so that $\psi(a) \in \mathbb{R}$. Hence, $\psi(a^*) = \overline{\psi(a)}$ for self-adjoint elements. Let $b \in \mathcal{A}$, then $b = b_1 + ib_2$ in which $b_1, b_2 \in \text{sa}(\mathcal{A})$. Since $\psi(b_1), \psi(b_2) \in \mathbb{R}$,

$$\psi(b^*) = \psi(b_1 - ib_2)$$

$$= \psi(b_1) - i\psi(b_2)$$

$$= \overline{\psi(b_1)} + i\overline{\psi(b_2)}$$

$$= \overline{\psi(b_1 + ib_2)}$$

$$= \overline{\psi(b)}$$

Hence, $\Gamma$ preserves $^*$ and so $\Gamma(\mathcal{A})$ is self-adjoint.

Towards isometry, we claim that for $a \in \text{sa}(\mathcal{A})$, $\|a^{2^n}\| = \|a\|^{2^n}$. For the base case, $\|a\|^2 = \|a^*a\| = \|aa\| = \|a\|^2$. Now, assume true for $n$ and since the involution is multiplicative for self-adjoint elements,

$$\|a\|^{2^{n+1}} = (\|a\|^{2^n})^2$$

$$= (\|a^{2^n}\|)^2$$

$$= \|(a^{2^n})^*a^{2^n}\|$$

$$= \|a^{2\cdot2^n}\|$$

$$= \|a^{2^n+1}\|$$

Now, combining the above with two previous results, and the fact that for a converging sequence of real numbers, every subsequence converges
to the same limit,
\[
\| \hat{a} \|_\infty = r(a) \\
= \lim_{n \to \infty} \| a^n \|^{1/n} \\
= \lim_{n \to \infty} (\| a^{2^n} \|)^{1/(2^n)} \\
= \lim_{n \to \infty} (\| a \|^{2^n})^{1/(2^n)} \\
= \lim_{n \to \infty} \| a \| = \| a \|
\]

Now, for any \( b \in A \), recall that \( b^* b \in \text{sa}(A) \). Since \( \Gamma \) is a \(*\)-homomorphism and that \( C(\Psi) \) itself is a \( C^* \)-algebra and satisfies the \( C^* \)-identity,
\[
\| b \|^2 = \| b^* b \| \\
= \| \hat{b} \hat{b} \|_\infty \\
= \| \hat{b} \hat{b} \|_\infty \\
= \| \hat{b} \|^2_\infty
\]

Thus, by taking roots, we have that \( \Gamma \) is an isometry. And, by the discussion at the start of the proof, this finishes the proof since we showed that \( \Gamma(A) \) is closed and dense, but in the process showed all other requirements for \( \Gamma \) to be an isometric \(*\)-isomorphism.

The following corollary will be used as a lemma in the result for the non-unital case.

**Corollary 2.2.0.2.** For a compact Hausdorff space \( X \), the maximal ideals of \( C(X) \) are all of the form \( M_x = \{ f \in C(X) : f(x) = 0 \} \) for every \( x \in X \).

**Proof.** \( C(X) \) is a unital commutative \( C^* \)-algebra, so by the previous result, \( C(X) \) is isometrically \(*\)-isomorphic to \( C(\Psi) \) where \( \Psi \) is the Maximal Ideal Space of \( C(X) \). Thus, there exists some homeomorphism \( h : \Psi \to X \). Let \( M \) be a maximal ideal of \( C(X) \), then there exists a unique \( \psi_M \in \Psi \) such that \( \ker \psi_M = M \). We show that \( M = M_{h(\psi_M)} \).

Let \( f \in M \), then \( \psi_M(f) = 0 \) so that \( \hat{f}(\psi_M) = 0 \), which implies via the homeomorphism and that \( \Gamma \) is an onto isometry and using \( \Gamma^{-1} \), \( f(h(\psi_M)) = 0 \). All these statements work in the opposite direction as well. Thus, we have equality and it follows that \( M \) is a Maximal Ideal as it is the kernel of \( \psi_M \) of the desired form.

Now, let \( x \in X \). Then \( h^{-1}(x) = \phi \in \Psi \). Let \( M = \ker \phi \). We show that \( M = M_x \). But, this follows the same argument since \( f(x) = 0 \iff \hat{f}(\phi) = \phi(f) = 0 \) since \( \phi \) corresponds to \( x \) via the homeomorphism and \( f \) corresponds to \( \hat{f} \) via \( \Gamma \).
Note that this result is also proven in [1] on page 219 and 220 using the Reisz Representation Theorem. The following lemma also comes from [1] as exercise 12 on page 67.

**Lemma 2.2.0.5.** If $X$ is locally compact Hausdorff and $X_\infty = X \cup \{\infty\}$ its one-point compactification, then $C_0(X)$ is isometrically *-isomorphic to the maximal ideal $M_\infty$ of $C(X_\infty)$, so that $C_0(X)$ are the continuous functions which vanish at $\infty$.

**Proof.** Define the map $\kappa : C_0(X) \rightarrow M_\infty$, by $\kappa(f) = \tilde{f}$, where we extend $f$ by $\tilde{f} = f \cdot \chi_X$. Note that $\chi_X$ is the characteristic function of $X \subset X_\infty$, so that $\tilde{f}$ agrees with $f$ on $X$ and $\tilde{f}(\infty) = 0$. It is clear that this extension is unique. Thus, one we show the map is well-defined, we will already have an injection.

Assume $f \in C_0(X)$. We show $\tilde{f} \in C(X_\infty)$ since it is clear that once we have this $\tilde{f} \in M_\infty$. Let $U \subset \mathbb{C}$ be open. If $0 \notin U$, then by definition of $\tilde{f}$, $\tilde{f}^{-1}(U) = f^{-1}(U)$, which is open in $X$ by continuity of $f$ and therefore open in $X_\infty$.

Next, assume that $0 \in U$, so that $\infty \in \tilde{f}^{-1}(U)$. Thus, we must show that $X_\infty \setminus \tilde{f}^{-1}(U)$ is compact. Since $U$ is open, we may find an $r > 0$ such that there is an open ball about 0 contained in $U$, i.e. $V = \{z : |z| < r\} \subset U$. Thus, we may write, $\tilde{f}^{-1}(U) = \{x \in X_\infty : \tilde{f}(x) \in V\} \cup \{x \in X : f(x) \in U \setminus V\}$. The latter set only considers $X$ and $f$ since $0 \notin U \setminus V$. Similar reasoning applies next since $\infty \notin X_\infty \setminus \tilde{f}^{-1}(U)$.

Therefore,

$$X_\infty \setminus \tilde{f}^{-1}(U) = \{x \in X : f(x) \in V\} \cap \{x \in X : f(x) \notin U \setminus V\}$$

$$= \{x \in X : f(x) \in V^c\} \cap \{x \in X : f(x) \in U^c \cup V\}$$

$$= \{x \in X : |f(x)| \geq r\} \cap \{x \in X : f(x) \in U^c\}$$

$$= \{x \in X : |f(x)| \geq r\} \cap f^{-1}(U^c)$$

The fact that we could move from $U^c \cup V$ to $U^c$ comes the fact that in the intersection we only also consider values in $V^c$, so in the intersection values form $U^c \cup V$ must only come from $U^c$. Let we have a contradiction. The first set is compact by definition of $C_0(X)$, and the second set is closed by continuity and that $U^c$ is closed. Thus, the intersection is closed and contained in the first compact set, and so the intersection is compact, which show that $\tilde{f} \in M_\infty$. Therefore, $\kappa$ is well-defined and injective by uniqueness. Injectivity could also come from the fact that this map is isometric since we introduce a 0 on a previously non-existent value.

Now, for onto, let $f \in M_\infty$. We show that $f|_X \in C_0(X)$ since clearly $\kappa(f|_X) = f$. Let $\varepsilon > 0$. Let $U = \{x \in X_\infty : |f(x)| < \varepsilon\} = f^{-1}(B_\varepsilon(0))$.
is open by continuity of \( f \) and \( \infty \in U \). Thus, by definition of one-point compactification, \( X \setminus U \) is compact. Now, \( \infty \notin X \setminus U \) and so, \( \{ x \in X : f(x) \notin U \} \) is compact. Thus, since \( \varepsilon > 0 \) was arbitrary and we only needed to consider values in \( X, f|_X \in C_0(X) \). Hence, \( \kappa \) is an isometric bijection. The inverse map is the restriction. The rest of the conditions (*-homomorphism) are routine since we only introduce a 0 on a previously non-existent value. Also, maximality comes from the previous result. □

The next lemma will allow us to make sense of \( C_0(\Psi) \) where \( \Psi \) is defined on a non-unital commutative \( C^* \)-algebra.

Lemma 2.2.0.6. If \( A \) is a non-unital commutative \( C^* \)-algebra, then \( \Psi \) is locally compact Hausdorff and \( \tilde{\Psi} \) is the one-point compactification of \( \Psi \) where \( \tilde{\Psi} \) is maximal ideal space of \( \tilde{A} \), the unitalization of \( A \).

Proof. Now, \( A \) may be identified with \( \tilde{A} \) as \( \{(a,0) : a \in A\} \). This also identifies \( A \) as a maximal ideal of \( \tilde{A} \). Since if \( A \) were properly contained in an ideal, \( I \), then it would have to have an element of the form \((a,\mu)\) such that \( \mu \neq 0 \). But, since \( I \) contains \( A \), \((a,0) \in I \). But, ideals are closed under addition, and so \( (0,\mu) = (a,\mu) - (a,0) \in I \). By absorption, \((0,1) = (0,\mu^{-1})(0,\mu) \in I \). Thus, \( I \) contains the identity is therefore all of \( \tilde{A} \).

\( A \) is a maximal ideal of \( \tilde{A} \), a unital commutative \( C^* \)-algebra, and so there exists a unique \( \psi_0 \in \tilde{\Psi} \) such that \( A = \ker \psi_0 \). Consider \((a,\mu)\), since \( \psi_0((0,1)) = 1 \),

\[
\psi_0((a,\mu)) = \psi_0((a,0) + (0,\mu)) = \psi_0((a,0)) + \psi_0((0,\mu)) = 0 + \psi_0(\mu(0,1)) = \mu \psi_0(0,1) = \mu \cdot 1 = \mu
\]

This defines \( \psi_0 \) on all of \( \tilde{A} \).

Define the map \( M : \Psi \to \tilde{\Psi} \) by \( M(\psi) = \tilde{\psi} \) where \( \tilde{\psi} \) is an extension of \( \psi \) to \( \tilde{A} \). We show that \( M \) is a homeomorphism onto \( \tilde{\Psi} \setminus \{\psi_0\} \). This would prove that \( \Psi \cup \{\psi_0\} = \tilde{\Psi} \) (the one-point compactification) since \( M \) would be an open embedding of \( \Psi \) into \( \tilde{\Psi} \) (a compact space) such that \( \tilde{\Psi} \setminus M(\Psi) = \{\psi_0\} \). Furthermore, since \( \tilde{\Psi} \) is Hausdorff, \( \Psi \) would be locally compact Hausdorff.

Now, for well-defined and injectivity, assume that \( \psi \in \Psi \). If \( \psi \) were to have an extension to \( \tilde{\Psi} \), call it \( \tilde{\psi} \), then by definition of extension,
its restriction to $A$ would be $\psi$, i.e. $\tilde{\psi}((a,0)) = \psi(a)$. Hence, for any $(a,\mu)$,
\[
\tilde{\psi}((a,\mu)) = \tilde{\psi}((a,0) + (0,\mu)) \\
= \psi((a,0)) + \tilde{\psi}((0,\mu)) \\
= \psi(a) + \mu
\]

Now, it is routine to check that $\tilde{\psi}$ is a non-zero complex homomorphism on $\tilde{A}$ since it must be non-zero for some $(a,0)$ by $\psi$ non-zero. And, by the approach we took, it is unique.

Assume by way of contradiction that there is some $\psi \in \Psi$ such that $M(\psi) = \psi_0$. Then, for $(a,\mu)$,
\[
\mu = \psi_0((a,\mu)) \\
= \tilde{\psi}((a,\mu)) \\
= \psi(a) + \mu
\]

Since, $a$ was arbitrary this would imply that $\psi(a) = 0$ for all $a$, a contradiction. Thus, $\psi_0 \notin M(\Psi)$.

For onto, we let $\psi \in \overline{\Psi} \setminus \{\psi_0\}$ and show that $M(\psi|_A) = \psi$. But, we must first verify that $\psi|_A \in \Psi$. It is easy to see that it is still a complex homomorphism since $(a,0)(b,0) = (ab,0)$. So, we prove that it is non-zero. Assume by way of contradiction that $\psi|_A$ is the zero homomorphism. Thus, $\ker \psi_0 = A \subseteq \ker \psi$. Since $\psi \neq \psi_0$, $\ker \psi_0$ is strictly contained in $\ker \psi$. Hence, there exists some $\mu \neq 0$ such that $\psi((a,\mu)) = 0$. Hence, applying the same trick as before,
\[
0 = \psi((a,\mu)) \\
= \psi((a,0)) + \mu \psi((0,1)) \\
= \psi|_A(a) + \mu \\
= 0 + \mu,
\]
a contradiction. To finish onto,
\[
\overline{\psi|_A((a,\mu))} = \psi|_A(a) + \mu \\
= \psi((a,0)) + \mu \psi((0,1)) \\
= \psi((a,0) + (0,\mu)) \\
= \psi((a,\mu))
\]

Thus, $M$ is a bijection and from the work shown its inverse is the restriction map. Also, this completes that $M(\Psi) = \overline{\Psi} \setminus \{\psi_0\}$. Now, let
\( \psi_\lambda \to \psi \) be a converging net in \( \Psi \). Fix, \((a, \mu)\), then
\[
M(\psi_\lambda)((a, \mu)) = \tilde{\psi}_\lambda((a, \mu)) = \psi_\lambda(a) + \mu \to \psi(a) + \mu = M(\psi)((a, \mu)).
\]
Since \((a, \mu)\) arbitrary, \(M\) is continuous. And, it is clear that converging nets are preserved through restriction, and so \(M^{-1}\) is continuous. Hence, \(M\) is a homeomorphism and by our beginning comments we are done. \(\square\)

Now, by the aid from this lemma, it’s time to extend the result of Theorem 2.2.0.3 to the non-unital case.

**Theorem 2.2.0.4.** If \(\mathcal{A}\) is a non-unital commutative \(C^*\)-algebra, then \(\mathcal{A}\) is isometrically \(*\)-isomorphic to \(C_0(\Psi)\), the continuous functions which vanish at \(\{\psi_0\}\).

**Proof.** \(\mathcal{A}\) is a maximal ideal of \(\widetilde{\mathcal{A}}\), its unitilization. But, \(\widetilde{\mathcal{A}}\) is isometrically \(*\)-isomorphic to \(C(\Psi) = C(\Psi \cup \{\psi_0\})\), by the previous lemma. Thus, \(\mathcal{A}\) is (via the Gelfand Map on \(\widetilde{\mathcal{A}}\)) a maximal ideal of \(C(\Psi \cup \{\psi_0\})\), and by another previous result \(\mathcal{A}\) is (again via the Gelfand Map) of the form \(M_\psi = \{(a, \mu) \in C(\Psi \cup \{\psi_0\}) : (a, \mu)(\psi) = 0\}\).

We now show that the \(\psi\) that defines \(M_\psi\) is \(\psi_0\). But, by definition of \(\psi_0\), for \((a, 0)\in \mathcal{A}\), \(\psi_0((a, 0)) = 0\). But, by the Gelfand Map, this implies that \((a, 0)(\psi_0) = 0\). Thus, since \((a, 0)\in \mathcal{A}\) was arbitrary, \(\mathcal{A}\) is isometrically \(*\)-isomorphic to \(M_{\psi_0}\), but since \(\psi_0\) forms the one-point compactification, by another previous result, \(\mathcal{A}\) is isometrically \(*\)-isomorphic to \(C_0(\Psi)\). \(\square\)

Next, we relate this back to the spectrum.

**Theorem 2.2.0.5.** If \(\mathcal{A}\) is a unital \(C^*\)-algebra and \(n\) is normal in \(\mathcal{A}\), then \(C^*(1, n)\) is isometrically \(*\)-isomorphic to \(C(\sigma(n))\). Moreover, \(n\) may be identified with \(z \in C(\sigma(n))\) such that \(z(\mu) = \mu\).

**Proof.** Let \(N = C^*(1, n)\). \(N\) is unital and commutative since \(n\) is normal. Thus, \(N\) is isometrically \(*\)-isomorphic to \(C(\Psi)\), where \(\Psi\) is the maximal ideal space of \(N\).

We must show that \(\Psi\) is homeomorphic to \(\sigma(n)\). Now, from the Gelfand Map \(\hat{n}\) is continuous on \(\Psi\), and since \(\sigma(n) = \{\psi(n) : \psi \in \Psi\}\), \(\hat{n} : \Psi \to \sigma(n)\) is well-defined and onto.

For injectivity, assume that \(\psi_1, \psi_2 \in \Psi\) and that \(\hat{n}({\psi}_1) = \hat{n}({\psi}_2)\) or \(\psi_1(n) = \psi_2(n)\). Thus, \(\overline{\psi_1(n)} = \overline{\psi_2(n)}\). But, by the proof of the unital commutative Gelfand-Naimark, this implied that \(\psi_1(\hat{n}) = \psi_2(\hat{n})\). Thus, by linear and multiplicative \(\psi_1(p(n, n^*)) = \psi_2(p(n, n^*))\). But, since these are continuous, by density of the polynomials, we have that \(\psi_1(m) = \psi_2(m)\) for all \(m \in N\). Hence \(\psi_1 = \psi_2\). Thus, \(\hat{n}\) is a continuous
bijection from a compact space to a Hausdorff space and is therefore a homeomorphism.

Thus, there is an isometric *-isomorphism, \( \Delta \), from \( C(\sigma(n)) \) to \( C(\Psi) \) such that \( \Delta(f)(\psi) = f(\hat{n}(\psi)) \). Thus, \( \Delta(z)(\psi) = z(\hat{n}(\psi)) = \hat{n}(\psi) \), so that \( \Delta(z) = \hat{n} \). Let \( \Lambda = \Gamma^{-1} \circ \Delta \). \( \Lambda(z) = \Gamma^{-1}(\Delta(z)) = \Gamma^{-1}(\hat{n}) = n \). Since, we simply composed isometric *-isomorphisms, we are done. \( \square \)

**Corollary 2.2.0.3.** If \( n \in A \) is normal in a non-unital commutative \( C^* \)-algebra, then \( C^*(n) \) is isometrically *-isomorphic to \( C_0(\sigma((n,0) \setminus \{0\})) \), the continuous functions which vanish at 0. And, we still have that \( n \) may be identified with \( z \).

**Proof.** \( C^*(n) \) as a non-unital commutative (by normality) \( C^* \)-algebra is isometrically *-isomorphic to \( C_0(\Psi) \), the continuous functions which vanish at \( \{\psi_0\} \). Now, \( 0 \in \psi((n,0)) \) since \( (n,0) \) is not invertible. Now, \( \Psi \cup \{\psi_0\} = \hat{\Psi} \), which is homeomorphic to \( \sigma((n,0)) \) via the homeomorphism \( (n,0) \). But, \( (n,0)(\psi_0) = \psi_0((n,0)) = 0 \). Thus, \( \psi_0 \) corresponds to 0 via the homeomorphism. Hence, since homeomorphisms are local homeomorphisms and that \( \Psi \) is open in \( \hat{\Psi} \) by definition of the one-point compactification topology, \( \Psi \) is homeomorphic to \( \sigma((n,0) \setminus \{0\}) \), where 0 is the one-point compactification of the latter space. Hence, \( C^*(n) \) is isometrically *-isomorphic to \( C_0(\sigma((n,0) \setminus \{0\})) \), the continuous functions, which vanish at 0.

From here, we refer to a restricted \( \Gamma \), which is discussed in the proof of 2.2.0.4, and construct a new \( \Lambda \) from the homeomorphisms between the locally compact hausdorff spaces. Providing us with the identification of \( n \) with \( z \). \( \square \)

This provides us with the Continuous Functional Calculus. Recall the \( \Lambda \) map defined at the end of the proof of **Theorem 2.2.0.5.** Since \( \Lambda(z) = n \). It follows by the fact that it is a *-homomorphism, \( \Lambda(p(z,z^*)) = p(n,n^*) \in C^*(n) \). And, by density of the polynomials in the continuous functions for \( f \in C(\sigma(n)) \) and continuity of \( \Lambda \), \( \Lambda(f(z)) = f(n) \in C^*(n) \). Thus, for any continuous function, \( f \in C(\sigma(n)) \), \( f(n) \in C^*(n) \). And, in the non-unital case by the previous corollary, it is further required that \( f \) vanishes at 0 for \( f(n) \in C^*(n) \).

**Notation 2.2.0.2.** In the sequel, we will refer to \( \Gamma, \Gamma^{-1}, \Lambda, \) and \( \Lambda^{-1} \) as the Gelfand map. This distinction comes down to the following. \( \Lambda \) is used when discussing the commutative \( C^* \)-algebras that are generated by a single normal element, \( \Gamma \) is used otherwise such as when we have an arbitrary commutative \( C^* \)-algebra.
Before we leave the realm of commutative $C^*$-algebras, we prove some facts about commutative subalgebras. We present a basic topological result.

**Proposition 2.2.0.4.** Let $(R, +, \cdot)$ be a $T_1$ topological ring. If $S$ is a commutative subring of $R$, then the closure of $S$, $\overline{S}$, is commutative.

**Proof.** Define $c(a, b) = a \cdot b - b \cdot a$. Since $\cdot$ and $+$ are continuous, then so is $c$. Since $R$ is $T_1$, $\{0\}$ is closed. Therefore, $c^{-1}(\{0\}) = \{(a, b) \in R \times R : a \cdot b = b \cdot a\}$ is closed. Note that since $S$ is commutative, $c^{-1}(\{0\}) \cap S \times S = S \times S$. Furthermore,

\[
\overline{S} \times \overline{S} = S \times S \\
= c^{-1}(\{0\}) \cap S \times S \\
\subseteq c^{-1}(\{0\}) \cap S \times S \\
= c^{-1}(\{0\}) \cap \overline{S} \times \overline{S}.
\]

But, this implies that $\overline{S} \times \overline{S} \subseteq c^{-1}(\{0\})$. Thus, for all $a, b \in \overline{S}$, $a \cdot b = b \cdot a$. \[\square\]

As any $C^*$-algebra is a $T_1$ topological ring, we have immediately that.

**Corollary 2.2.0.4.** Let $A$ be a $C^*$-algebra. If $B$ is a commutative subring of $A$, then $\overline{B}$ is commutative.

3. Positivity

The functional calculus is quite powerful in that it provides a abstract notion of positivity that reflects the concrete notion of positivity in $B(H)$. As expected, this requires a better grasp of the spectrum in this setting.

**Proposition 3.0.0.5.** Let $A$ be a $C^*$-algebra.

1. If $n \in A$ is normal, then $\|n\| = r(n)$.
2. If $n \in A$ is self-adjoint, then $\sigma(n) \subseteq \mathbb{R}$.
3. If $u \in A$ is unitary, then $\sigma(u)$ is contained in the unit circle.

**Proof.** For the sake of notation, we assume that $A$ is unital.

1. Since the Gelfand map is an isometry, if we let $z$ denote $z(x) = x$, then by **Theorem 2.2.0.5**

\[
\|n\| = \|z\|_{C(\sigma(n))} = \sup_{\mu \in \sigma(n)} |z(\mu)| = \sup_{\mu \in \sigma(n)} |\mu| = r(n).
\]
(2) Since the Gelfand map is self-adjoint, we have that \( n \mapsto z \) and \( n^* \mapsto \overline{z} \) implies that \( z(\mu) = \overline{z}(\mu) \) for all \( \mu \in \sigma(n) \). This provides that \( \mu = \overline{\mu} \) for all \( \mu \in \sigma(n) \). Thus \( \sigma(n) \subseteq \mathbb{R} \).

(3) We refer to the unit of \( A \) as \( e \). Thus,

\[ uu^* = u^*u = e, \]

and by the Gelfand map this translates to

\[ z\overline{z}(\mu) = \overline{z}z(\mu) = 1(\mu). \]

Thus, \( |z|^2 = 1_{C(\sigma(u))} \), which implies that \( |\mu|^2 = 1 \) for all \( \mu \in \sigma(u) \), so that \( \sigma(u) \) is contained in the unit circle. \( \square \)

4. Ideals and Quotients

4.1. Approximate Identity. These objects are crucial in the proof that every Ideal of a \( C^* \)-algebra is a \( C^* \)-algebra, which is necessary for the notion of an Isomorphism Theorem for \( C^* \)-algebra Theory.

Also, the object is useful for reproducing the GNS construction for non-unital \( C^* \)-algebras.

Furthermore, in the area of Noncommutative Metric Geometry and specifically in the area of locally compact metric spaces, we assume separability which provides us with a sequence versus a net, which further enhances the properties of the approximate identity.

Remark 4.1.0.1. In a non unital \( C^* \)-algebra, \( A \), we define \( \sigma(x) \) over the unitilization of \( A \), \( \tilde{A} \).

Definition 4.1.0.5. An approximate identity of a \( C^* \)-algebra is a net \( \{e_\mu\}_{\mu \in \Delta} \) on some directed set \( (\Delta, \leq) \) that satisfies the following properties.

i. \( e_\mu \geq 0 \) for all \( \mu \in \Delta \)

ii. \( \|e_\mu\| \leq 1 \) for all \( \mu \in \Delta \)

iii. \( \lambda \leq \mu \) implies \( e_\lambda \leq e_\mu \)

iv. \( \lim_{\mu} \|xe_\mu - x\| = \lim_{\mu} \|e_\mu x - x\| = 0. \)

Example 4.1.0.1. Consider \( C_0(\mathbb{R}) \). Notice that this \( C^* \)-algebra does not have a unit since it could not have constant functions by the requirement for the functions to vanish. If,

\[
e_n(x) = \begin{cases} 
0 & : \quad x \leq -(n+1) \\
x + n + 1 & : \quad -(n+1) < x \leq -n \\
1 & : \quad -n < x \leq n \\
-x + n + 1 & : \quad n < x \leq n + 1 \\
0 & : \quad n + 1 < x
\end{cases}
\]
Visually, these maps look like trapezoids of height 1 whose top and bottom base increase uniformly about the origin as $n$ increases. It is easy to check that $\{e_n\}_{n \in \mathbb{N}}$ is an approximate identity for our non unital $C^*$-algebra.

We prove some lemmas before our two main results. And, for the lemmas, we assume that $A$ is unital, with unit $e$. Since in the proof of our main theorems, the following lemmas will only come into use when we work within $\tilde{A}$.

**Lemma 4.1.0.7.** If $x \leq y$ and $z \in A$, then $z^*xz \leq z^*yx$.

**Proof.** By assumption, $y - x$ is positive. Thus, let $w$ be its positive square root. That is $w^2 = y - x$. Therefore,

\[
    z^*yz - z^*xz = z^*(y - x)z \\
    = z^*w^2z \\
    = z^*wzw \\
    = (wz)^*(wz)
\]

But, $x^*x \geq 0$ for all $x \in A$. Thus, $z^*yz - z^*xz \geq 0$. \qed

**Lemma 4.1.0.8.** Let $z \in A$. $z^*z \leq e \iff \|z\| \leq 1$.

**Proof.** First, the backwards direction. If $\|z\| \leq 1$, the $\|z\|^2 \leq 1$. By the $C^*$ identity, $\|z^*z\| \leq 1$. Since $z^*z \geq 0$, $r(z^*z) = \|z^*z\| \leq 1$. Thus, $\sigma(z^*z) \subseteq [0,1]$. Consider the continuous function $f(x) = 1 - x$. $f([0,1]) \subseteq [0,1]$. Hence, by the Spectral Mapping Theorem,

\[
    \sigma(e - z^*z) = \sigma(f(z^*z)) = f(\sigma(z^*z)) \subseteq [0,1].
\]

Thus, since $e - z^*z \in sa(A)$, and we just showed that its spectrum is positive, $e - z^*z \geq 0$.

Now, for the backwards direction, we assume that $e - z^*z \geq 0$, and so $\sigma(e - z^*z) \subseteq [0,\infty)$. Using the same $f$ as above, we have that $f([0,\infty)) \subseteq (-\infty,1]$ and that $f(e - z^*z) = z^*z$. Therefore, by the Spectral Mapping Theorem,

\[
    \sigma(z^*z) = \sigma(f(e - z^*z)) = f(\sigma(e - z^*z)) \subseteq (-\infty,1].
\]

But, $z^*z \geq 0$ implies that $\sigma(z^*z) \subseteq [0,1]$. Thus, since $z^*z \in sa(A)$, $\|z^*z\| = r(z^*z) \leq 1$. And, using the $C^*$ identity, $\|z\| \leq 1$. \qed

**Lemma 4.1.0.9.** If $x, y \in \text{Inv}(A)$ such that $x, y \geq 0$ and $x \leq y$, then $y^{-1} \leq x^{-1}$.
Proof. Denote the positive square roots of $x, y$ by $x^{1/2}, y^{1/2}$ respectively. Since $x, y \in Inv(\mathcal{A})$ some calculation shows that $x^{1/2}, y^{1/2}$ are invertible and their inverses can be denoted $x^{-1/2}, y^{-1/2}$, respectively, which are also in $sa(\mathcal{A})$, and the exponent laws apply. By Lemma 4.1.0.7,

$$y^{-1/2}xy^{-1/2} \leq y^{-1/2}yy^{-1/2} = y^{-1/2}y^{1/2}y^{1/2}y^{-1/2} = e.$$  

Let $z = x^{1/2}y^{-1/2}$, and so, $z^* = y^{-1/2}x^{1/2}$ and $z^*z = y^{-1/2}xy^{-1/2}$, which is the LHS of the above inequality. Thus, by Lemma 4.1.0.8, $||x^{1/2}y^{-1/2}|| = ||z|| \leq 1$. But, this also, implies that $||z^*|| \leq 1$, which again by Lemma 4.1.0.8, provides that $zz^* = (z^*)^*z^* \leq e$. But, $zz^* = x^{1/2}y^{-1/2}y^{-1/2}x^{1/2} = x^{-1/2}y^{-1}x^{1/2}$.

Thus, $x^{1/2}y^{-1}x^{1/2} \leq e$, and so by Lemma 4.1.0.7 and since $x^{-1/2} \in sa(\mathcal{A})$,

$$y^{-1} = x^{-1/2}(x^{1/2}y^{-1}x^{1/2})x^{-1/2} \leq x^{-1/2}(e)x^{-1/2} = x^{-1}$$

We are now ready to prove our main results, so as mentioned previously, we assume that $\mathcal{A}$ does not contain a unit and refer to its unitilization $\tilde{\mathcal{A}}$ when necessary.

**Theorem 4.1.0.6.** Every $C^*$-algebra contains an approximate identity.

**Proof.** Let $\mathcal{A}$ denote our $C^*$-algebra. Let $(\Delta, \preceq)$ be the directed set of finite subset of self-adjoint elements of $\mathcal{A}$ ordered by $\preceq$. Let $e$ denote the unit of $\tilde{\mathcal{A}}$.

Fix $\delta \in \Delta$, then $\tilde{\delta} = \{x_1, \ldots, x_n\}$. By spectral mapping with $\hat{p}(x) = x^2$, $x_1^2, \ldots, x_n^2 \geq 0$, and that the positive elements of $\mathcal{A}$ form a positive cone, $x_\delta = \sum_{i=1}^n x_i^2 \geq 0$.

Now, by functional calculus, if $a \in sa(\mathcal{A})$, $C_0(\sigma(a))$ is the space of continuous functions vanishing at 0. Furthermore, if $f \in C_0(\sigma(a))$, $f(a) \in C^*(a)$ with $f(a) \in sa(\mathcal{A})$.

If $f_n(t) = nt(1 + nt)^{-1}$, then $f_n \in C_0(\sigma(x_\delta))$ since positive elements are self-adjoint. Now, by spectral mapping, since the image of $f_n$ is bounded by $[0, 1]$ on $[0, \infty)$, $\sigma(f_n(x_\delta)) = f_n(\sigma(x_\delta)) \subseteq [0, 1]$. But, $f_n(x_\delta) \in sa(\mathcal{A})$, and so $f_n(x_\delta) \geq 0$. Let $e_\delta = f_n(x_\delta)$. Then, $e_\delta \geq 0$ and since the spectral radius agrees with the norm of self-adjoint elements $||e_\delta|| \leq 1$. We show that $\{e_\delta\}_{\delta \in \Delta}$ is an approximate identity, and note that we have already shown properties i., ii.
Now, for property iii., assume that $\lambda \leq \mu$ so that $\lambda = \{x_1, \ldots, x_n\}, \mu = \{x_1, \ldots, x_m\}$ where $n \leq m$. We show that $e_\lambda \leq e_\mu$. Note the following,

$$e_\lambda \leq e_\mu \iff 0 \leq e_\mu - e_\lambda$$
$$\iff 0 \leq -e_\lambda + e - e_\lambda$$
$$\iff 0 \leq (e - e_\lambda) - (e - e_\mu)$$
$$\iff e - e_\mu \leq e - e_\lambda$$

Thus, we show the last statement in the string of equivalences. Notice that $1 - f_k(t) = (1 + kt)^{-1}$. And, so $e - e_\mu = e - f_n(x_\mu) = (e + nx_\mu)^{-1}$. And, similarly, $e - e_\lambda = (e + mx_\lambda)^{-1}$. Further, note that by the fact that positive elements form a positive cone and that squares of self-adjoint elements are positive, since $n - m \geq 0$, $0 \leq (n - m) \sum_{i=1}^{m} x_i^2 + nx_{m+1}^2 + \cdots + nx_n^2$. But, the RHS is the same as $nx_\mu - mx_\lambda$. Thus, by Lemma 4.1.0.9,

$$e - e_\mu \leq e - e_\lambda \iff (e + nx_\mu)^{-1} \leq (e + mx_\lambda)^{-1}$$
$$\iff e + mx_\lambda \leq e + nx_\mu$$
$$\iff 0 \leq e + nx_\mu - (e + mx_\lambda) = nx_\mu - mx_\lambda$$

Hence, since the last expression holds true, $e - e_\mu \leq e - e_\lambda$. And, therefore, $e_\lambda \leq e_\mu$ if $\lambda \leq \mu$.

For part iv., we only prove that $\|x - xe_\lambda\| \to 0$ for $x \in sa(A)$ since the involution is an isometry and that $(x - xe_\lambda)^* = x - e_\lambda x$. Also, every element in a C*-algebra is the linear combination of two elements in $sa(A)$. Along with continuity of the norm over addition and scalar multiplication, we would be done. Let $x \in sa(A)$, for convergence in net, we must show for every $\varepsilon > 0$, there exists $\mu \in \Delta$ such that for every $\lambda \in \Delta$ with $\mu \leq \lambda$, $\|x - xe_\lambda\| < \varepsilon$.

Let $x \in sa(A), \varepsilon > 0$. There exists $m \in \mathbb{N}$ such that $1/4m < \varepsilon$. Choose $\mu \in \Delta$ such that $x \in \mu$ and $|\mu| = m$, let $\mu \leq \lambda$ so that $\lambda = \{x_1, \ldots, x_n\}$ with $m \leq n$. Since $x \in \lambda$, $x^2 \leq x_\lambda$. By Lemma 4.1.0.7, since $e - e_\lambda \in sa(A)$,

$$(e - e_\lambda)x^2(e - e_\lambda) \leq (e - e_\lambda)x_\lambda(e - e_\lambda).$$

Now, if $g_n(t) = (1 - f_n(t))t(1 - f_n(t)) = t(1 + nt)^{-2}$, then $g_n$ is a continuous function vanishing at 0 and the RHS of the above inequality is $g_n(x_\lambda)$. But, on $[0, \infty)$, $g_n$ is bounded above by $1/4n$. But, $\sigma(x_\lambda) \subseteq [0, \infty)$. Thus, by spectral mapping $\sigma(g_n(x_\lambda)) \subseteq [0, 1/4n]$. Consider $p(x) = 1/4n - x$, $p([0, 1/4n]) \subseteq [0, 1/4n]$, then again by spectral mapping,

$$\sigma(e/4n - g_n(x_\lambda)) = p(\sigma(g_n(x_\lambda))) \subseteq [0, 1/4n].$$
Yet, \( e/4n - g_n(x_\lambda) \in sa(A) \). Hence, \( e/4n - g_n(x_\lambda) \geq 0 \). Therefore, \( g_n(x_\lambda) \leq e/4n \). Thus,
\[
(e - e_\lambda)x^2(e - e_\lambda) \leq g_n(x_\lambda) \leq e/4n.
\]

Furthermore, the LHS is of the form \( z^*z \) where \( z = x(e - e_\lambda) = x - x e_\lambda \). Thus, \( z^*z \leq e/4n \), and by a similar argument in the proof of Lemma 4.1.0.8, \( \|z\| \leq 1/4n \). Therefore, since \( m \leq n \),
\[
\|x - x e_\lambda\| = \|z\| \leq 1/4n \leq 1/4m < \varepsilon.
\]

\[\square\]

It can be the case that one does require a net and cannot use a sequence. For if we consider the compact operators \( K(H) \) on a non-separable Hilbert space, then there does not exist an approximate identity which is a sequence. In the separable case, there is a sequence of finite rank operators that satisfy the approximate identity requirements which differs than the canonical one constructed in the proof of Theorem 4.1.0.6.

But, for our purposes in Noncommutative Metric Geometry, we deal mainly with separable \( C^* \)-algebras and have the following useful result, which becomes even more beneficial in the treatment of Bounded-Lipschitz Distances on the State Space of a \( C^* \)-algebra [4] and therefore of Quantum Locally Compact Metric Spaces [5].

**Corollary 4.1.0.5.** Every separable \( C^* \)-algebra contains an approximate identity which is a sequence.

**Proof.** Let \( Q \) be the countable dense subset of our \( C^* \)-algebra, \( A \). Denote the countable dense subset of \( sa(A) \), by \( \{y_n\}_{n=1}^\infty := Q \cap sa(A) \). As in the proof of Theorem 4.1.0.6, we work in \( sa(A) \) for the same reasons. Define \( e_n := e_{(y_1,\ldots,y_n)} \). Then, by the proof of Theorem 4.1.0.6, \( \{e_n\}_{n=1}^\infty \) satisfy properties i., ii., and iii.

Let \( \varepsilon > 0 \) and \( x \in sa(A) \). There exists \( y_p \) such that \( \|x - y_p\| < \varepsilon/3 \). Choose \( N_1 \in \mathbb{N} \) such that \( p \leq N_1 \). Also, choose \( N_2 \in \mathbb{N} \) such that \( 1/4N_2 < \varepsilon/3 \). Let \( N = \max\{N_1,N_2\} \). Now, notice that for \( n \geq N \),
\[
\|y_p - e_n y_p\| \leq 1/4N
\]
since we satisfy the same conditions which produced this inequality in the proof of Theorem 4.1.0.6. The conditions being that \( p \leq N \) implies that \( y_p \in \{y_1,\ldots,y_n\} \) and that \( e_n = e_{(y_1,\ldots,y_n)} \).

Thus, if \( n \geq N \),
\[
\|x - e_n x\| \leq \|x - y_p\| + \|y_p - e_n y_p\| + \|e_n y_p - e_n x\|
\]
\[
< \varepsilon/3 + 1/4N + \|e_n\| \cdot \|y_p - x\|
\]
\[
< \varepsilon/3 + \varepsilon/3 + 1 \cdot \varepsilon/3
\]
\[
= \varepsilon.
\]
And, we are done for all elements in $A$ by isometry of involution and continuity of addition and scalar multiplication since every element in a $C^*$-algebra is the linear combination of two elements in $sa(A)$. 

4.2. Ideals and Quotients are $C^*$-algebras.

**Theorem 4.2.0.7.** Ideals are $C^*$-Algebras.

*Proof.* Let $\mathcal{C}$ be a $C^*$-algebra, and let $I$ be an ideal of $\mathcal{C}$. Consider $J = I \cap I^*$, where $I^* = \{a^* : a \in I\}$. Now, $J$ is an ideal of $\mathcal{C}$ and is closed under adjoints. Therefore, $J$ is a $C^*$-algebra. The $\{j_\nu\}_{\nu \in \Delta}$ be an approximate identity for $J$. Now, let $a \in I$. We need to show that $a^* \in I$. Note that $a^*a \in J$, so $\|aa^*j_\nu - a^*\| \to 0$. And, since $j_\nu \in J \subset I$, by ideal $a^*j_\nu, j_\nu a^* \in I$ for all $\nu \in \Delta$. Thus, by the $C^*$ identity and the above statements along with the fact that $j_\nu$ are self-adjoint with $\|j_\nu\| \leq 1$,

$$\|a^*j_\nu - a^*\|^2 = \|(a^*j_\nu - a^*)^*(a^*j_\nu - a^*)\|$$

$$= \|(j_\nu a - a)(a^*j_\nu - a^*)\|$$

$$= \|j_\nu aa^*j_\nu - j_\nu aa^* - aa^*j_\nu + aa^*\|$$

$$\leq \|j_\nu aa^*j_\nu - j_\nu aa^*\| + \|aa^*j_\nu - aa^*\|$$

$$= \|j_\nu (aa^*j_\nu - aa^*)\| + \|aa^*j_\nu - aa^*\|$$

$$\leq \|j_\nu\| \cdot \|aa^*j_\nu - aa^*\| + \|aa^*j_\nu - aa^*\|$$

$$\leq 1 \cdot \|aa^*j_\nu - aa^*\| + \|aa^*j_\nu - aa^*\|$$

$$= 2\|aa^*j_\nu - aa^*\|$$

And, so

$$\|a^*j_\nu - a^*\|^2 \leq 2\|aa^*j_\nu - aa^*\| \to 0.$$ 

Therefore, $\lim_\nu \|a^*j_\nu - a^*\| = 0$, which implies that $\lim_\nu a^*j_\nu = a^*$. But, $\{a^*j_\nu\}_{\nu \in \Delta} \in I$ is a convergent net and since ideals are closed it must be the case that its limit, $a^* \in I$. Thus, $I$ is self-adjoint by arbitrary $a \in I$. Hence, it is a sub-$C^*$-algebra of $\mathcal{C}$ and is therefore itself a $C^*$-algebra. 

Now, let $\mathcal{C}$ be a possibly non-unital $C^*$-algebra with a non-zero proper ideal $I$. Then, it is easy to see that $\mathcal{C}/I$ is a Banach Algebra. Thus, the only difficulty remains is in showing that $\mathcal{C}/I$ satisfies the $C^*$-identity. Before we prove the second main result, we prove a useful lemma.

**Lemma 4.2.0.10.** For $a \in \mathcal{C}$, $\|[a]\| = \lim_\nu \|a - ai_\nu\|$, where $\{i_\nu\}_{\nu \in \Delta}$ is the approximate identity for $I$. 

Proof. Of course, such an approximate identity exits by the previous theorem. Now, since \( i, \nu \) is positive with norm \( \leq 1 \), with spectral mapping and \( p(x) = 1 - x \), it is clear that \( \|e - i, \nu\| \leq 1 \) where \( e \) is from the unitilization of \( C \). Let \( \varepsilon > 0 \). Now, recalling the definition of the quotient norm \( \|[a]\| = \inf_{i \in I} \|a - i\| \), we have that there exists \( i \in I \) such that \( \|a - i\| \leq \|[a]\| + \varepsilon/2 \). Also, note that since \( ai, \nu \in I \) by ideal, \( \|[a]\| \leq \|a - ai, \nu\| \) for all \( \nu \in \Delta \) by definition of quotient norm. Then,

\[
\|[a]\| \leq \|a - ai, \nu\| = \|a(e - e, \nu)\|
= \|a(e - e, \nu) - i(e - e, \nu) + i(e - e, \nu)\|
\leq \|(a - i)(e - e, \nu)\| + \|i(e - e, \nu)\|
\leq \|a - i\| \cdot \|e - e, \nu\| + \|i - i, e, \nu\|
\leq \|a - i\| \cdot 1 + \|i - i, e, \nu\|
< \|[a]\| + \varepsilon/2 + \varepsilon/2
= \|[a]\| + \varepsilon
\]

for \( \nu \) suitably far into the net. Thus, be squeeze theorem for nets,

\[
\|[a]\| = \lim_{\nu} \|a - ai, \nu\|.
\]

\( \square \)

**Theorem 4.2.0.8** (Segal). Quotients are \( C^* \)-algebras.

Proof. Using the previous lemma for both \( a \) and \( a^*a \), and the fact the \( C^* \)-identity is satisfied by \( \|\cdot\| \), we have that

\[
\|[a]\|^2 = \lim_{\nu} \|a(e - j, \nu)\|^2
= \lim_{\nu} \|(a(e - j, \nu))^*a(e - j, \nu)\|
= \lim_{\nu} \|(e - j, \nu)a^*a(e - j, \nu)\|
\leq \lim_{\nu} \|(e - j, \nu)\| \cdot \|a^*a(e - j, \nu)\|
\leq \lim_{\nu} 1 \cdot \|a^*a(e - j, \nu)\|
= \|[a^*a]\|
\]
Notice that the involution is still an isometry for $\mathcal{C}/I$, since $I$ is self-adjoint,

$$\|[a]\| = \inf_{i \in I} \|a - i\|$$

$$= \inf_{i \in I} \|(a - i)^*\|$$

$$= \inf_{i \in I} \|a^* - i^*\|$$

$$= \inf_{i \in I} \|a^* - i\|$$

$$= \|[a^*]\|$$

Thus, by the above and Banach Algebra, $\|[a^*a]\| \leq \|[a^*]\| \cdot \|[a]\| = \|[a]\| \cdot \|[a]\| = \|[a]\|^2$. Therefore, for all $a \in \mathcal{C}$,

$$\|[a^*a]\| = \|[a]\|^2.$$

Before we finish this section by proving a first isomorphism theorem for $C^*$-algebras, we must take care of one remaining issue. We would like in our first isomorphism theorem to only assume that our given map a $\ast$-homomorphism. This would be ideal and is in fact all we need. So, we prove the following lemmas.

**Lemma 4.2.0.11.** A $\ast$-homomorphism between $C^*$-algebras is contractive.

**Proof.** □

**Lemma 4.2.0.12.** An injective $\ast$-homomorphism between $C^*$-algebras is an isometry.

**Proof.** □

**Theorem 4.2.0.9** (First Isomorphism Theorem for $C^*$-algebras). If $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a $\ast$-homomorphism between $C^*$-algebras $\mathcal{A}, \mathcal{B}$, then $\varphi$ is an isometric $\ast$-isomorphism between the $C^*$-algebras $\mathcal{A}/\ker \varphi$ and $\varphi(\mathcal{A})$.

**Proof.** □

### 5. Irrational Rotation Algebra

6. $\ell^1(\mathbb{Z}^d)$ as a $C^*$-Algebra with Convolution

Recall:

$$\ell^1(\mathbb{Z}^d) := \left\{ f : \mathbb{Z}^d \rightarrow \mathbb{C} : \sum_{n \in \mathbb{Z}^d} |f(n)| < \infty \right\}$$
is a Banach Space with norm \( \|f\|_1 = \sum_{n \in \mathbb{Z}^d} |f(n)| \). We use this structure to develop a well-defined multiplication that satisfies the Banach Algebra bound on multiplication. Next, we define an involution and create a new norm which satisfies the \( C^* \) condition while still satisfying all the requirements for a Banach Algebra after completion. And, further, for \( \mathbb{Z}^d \), this \( C^* \)-Algebra will be unital and commutative.

For the remainder of this section 1, we denote \( \ell^1(\mathbb{Z}^d) \) as \( \ell^1 \).

6.1. Multiplication.

**Definition 6.1.0.6.** If \( f, g \in \ell^1 \), then the convolution of \( f, g \) is defined as
\[
(f * g)(k) = \sum_{n \in \mathbb{Z}^d} f(n)g(k-n).
\]

**Proposition 6.1.0.6.** Let \( f, g \in \ell^1 \), then \( (f * g) \in \ell^1 \) with \( \|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1 \)

**Proof.** Let \( f, g \in \ell^1 \). Then by applying Hölder’s Inequality,
\[
|(f * g)(k)| = \left| \sum_{n \in \mathbb{Z}^d} f(n)g(k-n) \right| \leq \|f\|_1 \cdot \|g\|_\infty < \infty
\]

Now, by absolute convergence, we may rearrange in the following manner,
\[
\|f * g\|_1 = \sum_{k \in \mathbb{Z}^d} |(f * g)(k)| = \sum_{k \in \mathbb{Z}^d} \left| \sum_{n \in \mathbb{Z}^d} f(n)g(k-n) \right|
\]
\[
\leq \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} |f(n)g(k-n)| = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} |f(n)| \cdot |g(k-n)|
\]
\[
= \sum_{n, k \in \mathbb{Z}^d} |f(n)||g(k-n)| = \sum_{n, m \in \mathbb{Z}^d} |f(n)||g(m)|
\]
\[
= \left( \sum_{n \in \mathbb{Z}^d} |f(n)| \right) \left( \sum_{m \in \mathbb{Z}^d} |g(m)| \right) = \|f\|_1 \cdot \|g\|_1
\]

Next, we verify that this multiplication produces an Algebra on \( \ell^1 \) with addition and scalar multiplication.

**Theorem 6.1.0.10.** \((\ell^1, +, *, \cdot)\) is a unital commutative Banach Algebra.

**Proof.** We only check: commutativity, distributivity, associativity, and unit, as \( \lambda(f * g) = ((\lambda f) * g) = (f * (\lambda g)) \) is clear along with the remaining vector space properties. Also, by Proposition 1 and that \( \ell^1 \) is a Banach Space, the Banach properties of the Algebra are satisfied.
Let $f, g, h \in \ell^1$

**Commutativity:** We only need to check that for arbitrary $k \in \mathbb{Z}^d$ that the function values commute.

$$(f \ast g)(k) = \sum_{n \in \mathbb{Z}^d} f(n)g(k - n) \quad \text{let } m = k - n,$$

$$= \sum_{m \in \mathbb{Z}^d} f(k - m)g(m) = \sum_{m \in \mathbb{Z}^d} g(m)f(k - m) = (g \ast f)(k)$$

Thus, $(f \ast g) = (g \ast f)$ for all $f, g \in \ell^1$.

**Distributivity:** Again, fix $k \in \mathbb{Z}^d$.

$$(f \ast (g + h))(k) = \sum_{n \in \mathbb{Z}^d} f(n)(g + h)(k - n) = \sum_{n \in \mathbb{Z}^d} [f(n)g(k - n) + f(n)h(k - n)]$$

$$= \sum_{n \in \mathbb{Z}^d} f(n)g(k - n) + \sum_{n \in \mathbb{Z}^d} f(n)h(k - n) = (f \ast g)(k) + (f \ast h)(k)$$

$$= (g \ast f)(k) + (h \ast f)(k) \quad \text{by commutativity}$$

$$= ((g + h) \ast f)(k) \quad \text{by a similar argument}$$

Thus, $(f \ast (g + h)) = (f \ast g) + (f \ast h) = (g \ast f) + (h \ast f) = (((g + h) \ast f)$

for all $f, g, h \in \ell^1$.

**Associativity:** Fix $k \in \mathbb{Z}^d$.

$$(f \ast (g \ast h))(k) = \sum_{n \in \mathbb{Z}^d} f(n)(g \ast h)(k - n) = \sum_{n \in \mathbb{Z}^d} f(n) \sum_{m \in \mathbb{Z}^d} g(m)h(k - n - m)$$

$$= \sum_{n, m \in \mathbb{Z}^d} f(n)g(m)h(k - (n + m))$$

$$( (f \ast g)\ast h))(k) = \sum_{p \in \mathbb{Z}^d} (f \ast g)(p)h(k - p) = \sum_{p \in \mathbb{Z}^d} \left(\sum_{q \in \mathbb{Z}^d} f(q)g(p - q)\right)h(k - p)$$

$$= \sum_{p, q \in \mathbb{Z}^d} f(q)g(p - q)h(k - p) \quad \text{Now, substitute } q = n \text{ and } p = m + n$$

$$= \sum_{n, m \in \mathbb{Z}^d} f(n)g(m)h(k - (n + m)) = (f \ast (g \ast h))(k)$$

$k$ arbitrary implies that $((f \ast g) \ast h)) = (f \ast (g \ast h))$.

**Unit:** Fix $k \in \mathbb{Z}^d$, and define $\delta_k \in \ell^1$ as:

$$\delta_k(n) = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$
Consider $\delta_0$, where $0 = (0, \ldots, 0) \in \mathbb{Z}^d$, and note that by definition $\delta_0(k-n) = 1 \iff n = k$, and 0 otherwise. Thus,

$$(f \ast \delta_0)(k) = \sum_{n \in \mathbb{Z}^d} f(n)\delta_0(k-n) = f(k)$$

Therefore, $(f \ast \delta_0) = f$, and $f = (\delta_0 \ast f)$ by commutativity. $\delta_0$ is the unit for $(\ell^1, +, \ast, \cdot)$, a unital commutative Banach Algebra. □

Next we introduce an involution, $\ast$, and discuss why it does not give us a $C^*$ algebra structure of $\ell^1$ with convolution under the 1-norm, and further develop a new norm, $\| \cdot \|_{C^*}$, constructed from representations of $\ell^1$ for which $(\ell^1, +, \ast, \ast, \cdot, \| \cdot \|_{C^*})$ is a $C^*$ algebra.

6.1.1. Involution and $\delta_k$.

**Definition 6.1.1.1.** Let $f \in \ell^1$ then we define the involution of $f$ pointwise as $f^*(k) = \bar{f}(-k)$.

Note that this is clearly an involution and is continuous since translation and conjugation are continuous.

**Lemma 6.1.1.1.**

1. $\forall m, j \in \mathbb{Z}^d \quad \delta_j \ast \delta_m = \delta_{j+m}$
2. $\forall j \in \mathbb{Z}^d \quad \delta_j$ is unitary with respect to the involution, $\ast$, and convolution, $\ast$.
3. $\forall m \in \mathbb{Z}^d \quad \exists m_1, \ldots, m_d \in \mathbb{Z}$ s.t.

$$\delta_m = \prod_{j=1}^d \delta_{e_j}^{m_j},$$

where $e_1, \ldots, e_d$ are the canonical basis for $\mathbb{Z}^d$, and the products and powers are respect to $\ast$

**Proof.**

1. Fix $m, j \in \mathbb{Z}^d$.

$$\delta_j \ast \delta_m(j + m) = \sum_{n \in \mathbb{Z}^d} \delta_j(n)\delta_m(j + m - n)$$

$$= \delta_j(j)\delta_m(j + m - j) = \delta_j(j)\delta_m(m) = 1$$

Thus, $(\delta_j \ast \delta_m)(j + m) = 1$ and 0 otherwise. Hence, $\delta_j \ast \delta_m = \delta_{j+m}$.

2. Fix $j \in \mathbb{Z}^d$. Now, $\delta_j^* = \delta_{-j}$. And, by part 1,

$$\delta_j^* \ast \delta_j = \delta_{-j} \ast \delta_j = \delta_{-j} \ast \delta_j = \delta_0.$$  

Also, by commutativity, we have that $\delta_j$ is unitary.
3. Let $m \in \mathbb{Z}^d$ such that $m = (m_1, \ldots, m_d)$, and $m_1, \ldots, m_d \in \mathbb{Z}^+$. Then,

$$m = \sum_{j=1}^{d} m_j e_j \quad \text{where} \quad m_1, \ldots, m_d \in \mathbb{Z}^+$$

Hence, by part 1,

$$\delta_m = \delta_{\sum_{j=1}^{d} m_j e_j} = \delta_{m_1 e_1} \cdots \delta_{m_d e_d}$$

But,

$$\delta_{m_1 e_1} \cdots \delta_{m_d e_d} = \underbrace{\delta_{e_1} \cdots \delta_{e_1}}_{m_1\text{-times}} \underbrace{\delta_{e_2} \cdots \delta_{e_d}}_{m_d\text{-times}}$$

Thus,

$$\delta_m = \prod_{j=1}^{d} \delta_{e_j}^{m_j},$$

where the products and powers are with respect to $\ast$. And, it is now clear how to extend this argument for all $m \in \mathbb{Z}^d$. \hfill \Box

From Lemma 1, $\mathcal{A} = \text{alg}\{\delta_{e_1}, \ldots, \delta_{e_d}\}$, is a dense subalgebra for $\ell^1$. Also, we may find a simple example of why $\ell^1$ is not a $C^*$ algebra with the 1-norm. Consider $f = \delta_k + \delta_0 - \delta_k$, where $0 \neq k$. Now, $\|f\|_1 = 3$. And, $f^* = \delta_k^* + \delta_0 - \delta_k$.

$$f^* \ast f = \delta_k^* \delta_k + \delta_k^* \delta_0 - \delta_k^* \delta_k + \delta_0 - \delta_0^* - \delta_k^* - \delta_2k - \delta_k + \delta_k^* - \delta_k$$

$$= \delta_{-k} \delta_k + \delta_{-k} \delta_{-k} + \delta_{-k} \delta_k + \delta_0 - \delta_0 + \delta_{-k} - \delta_{-2k} - \delta_k + \delta_k \delta_{-k} \text{ by } \delta_k^* = \delta_{-k}$$

$$= \delta_{-0} + \delta_{-k} - \delta_{-2k} - \delta_{-k} \delta_{-2k} - \delta_k + \delta_k$$

But, $\|f^* \ast f\|_1 = 5$. And, so $\|f^* \ast f\|_1 \neq \|f\|_1^2$. Next, we remedy this by using representations to develop a $C^*$ norm.

6.1.2. A Representation. Let $k, t \in \mathbb{Z}^d$, then recall that $\theta(k, t)$ is a bilinear skew-symmetric map into the integers.

**Theorem 6.1.2.1.** Fix $t \in \mathbb{Z}^d$, then $\pi_t : \ell^1 \to \mathbb{C}$ defined by

$$\pi_t(f) = \sum_{k \in \mathbb{Z}^d} f(k) e^{2\pi i \theta(k, t)}$$

is a representation of $\ell^1$.

**Proof.** Linearity is clear. Let $f, g \in \ell^1$.

**Multiplicative:**
\[
\pi_t(f * g) = \sum_{k \in \mathbb{Z}^d} (f * g)(k) e^{2\pi i \theta(k, t)}
\]
\[
= \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} f(n) g(k - n) e^{2\pi i \theta(k, t)} e^{2\pi i \theta(n, t)} \right)
\]
\[
= \sum_{k, n \in \mathbb{Z}^d} f(n) e^{2\pi i \theta(n, t)} g(k - n) e^{2\pi i \theta(k, t)} e^{2\pi i \theta(-n, t)}
\]
\[
= \sum_{k, n \in \mathbb{Z}^d} f(n) e^{2\pi i \theta(n, t)} g(k - n) e^{2\pi i \theta(k, t)}
\]
\[
= \left( \sum_{n \in \mathbb{Z}^d} f(n) e^{2\pi i \theta(n, t)} \right) \left( \sum_{k \in \mathbb{Z}^d} g(k - n) e^{2\pi i \theta(k, t)} \right) = \pi_t(f) \pi_t(g)
\]

**- homomorphism:**

\[
\pi_t(f^*) = \sum_{k \in \mathbb{Z}^d} (f^*)(k) e^{2\pi i \theta(k, t)}
\]
\[
= \sum_{k \in \mathbb{Z}^d} \overline{f(-k)} e^{2\pi i \theta(k, t)} = \sum_{k \in \mathbb{Z}^d} f(-k) e^{-2\pi i \theta(k, t)}
\]
\[
= \sum_{k \in \mathbb{Z}^d} f(-k) e^{-2\pi i \theta(k, t)} \quad \text{since conjugation is additive},
\]
\[
= \sum_{k \in \mathbb{Z}^d} f(-k) e^{2\pi i \theta(-k, t)} \quad \text{since } \theta \text{ is bilinear},
\]
\[
= \overline{\pi_t(f)} = \pi_t(f^*)
\]

Also, note that, \( \pi_t(f) = \hat{f}(t) \), the Fourier Transform of \( f \). \( \square \)

6.1.3. New Norm.

**Definition 6.1.3.1.** Define \( \| \cdot \|_{C^*} : \ell^1 \to \mathbb{R} \) by

\[
\|f\|_{C^*} = \sup\{||\pi(f)|| : \pi \text{ is a } * - \text{ representation}\}.
\]

**Theorem 6.1.3.1.** \( \|\cdot\|_{C^*} \) is a \( C^* \) norm for the completion of \( (\ell^1, +, *, *) \) under this norm.

**Proof.** Since by 6.1.2, we have the existence of a representation, the supremum in the definition is allowed. Also, Clearly, the norm satisfies positive homogeneity. Let \( f, g \in \ell^1 \).
Recall that if $\pi$ is a representation, then $\|\pi(f)\| \leq \|f\|$.

**Finite:** By the above statement,
$$\|f\|_{C^*} = \sup\{\|\pi(f)\| : \pi \text{ is a } * \text{- representation}\}$$
$$\leq \sup\{\|f\| : \pi \text{ is a } * \text{- representation}\}$$
$$= \|f\| < \infty$$

**Positive Definite:** If $f = 0 \in \ell^1$, then
$$\|0\|_{C^*} \leq \|0\| = 0.$$  

Now, assume $\|f\|_{C^*} = 0$. Hence, by 6.1.2,
$$0 = \|f\|_{C^*} = \sup\{\|\pi(f)\| : \pi \text{ is a } * \text{- representation}\}$$
$$\geq \|\pi_\ell(f)\| = \|\hat{f}(t)\|_1.$$  

Thus, $\hat{f}(t) = 0$, and since the Fourier Transform of $f$ is 0, $f = 0$ in $\ell^1$.

**Triangle Inequality:** Fix a representation $\rho$, then
$$\|\rho(f + g)\| = \|\rho(f) + \rho(g)\| \leq \|\rho(f)\| + \|\rho(g)\|$$
$$\leq \sup\{\|\pi(f)\| + \|\pi(g)\| : \pi \text{ is a } * \text{- representation}\}$$
$$= \sup\{\|\pi(f)\| : \pi \text{ is a } * \text{- representation}\} + \sup\{\|\pi(g)\| : \pi \text{ is a } * \text{- representation}\}$$
$$= \|f\|_{C^*} + \|g\|_{C^*}.$$  

This implies that $\|\rho(f + g)\| \leq \|f\|_{C^*} + \|g\|_{C^*}$, for an arbitrary representation $\rho$. Thus,
$$\|f + g\|_{C^*} = \sup\{\|\pi(f + g)\| : \pi \text{ is a } * \text{- representation}\}$$
$$\leq \|f\|_{C^*} + \|g\|_{C^*}.$$  

Therefore, $\|\cdot\|_{C^*}$ is a norm on $\ell^1$. From now on, we only consider $\ell^1$ as the completion of $\ell^1$ under $\|\cdot\|_{C^*}$.

**Banach Algebra:** By the previous statement, $\ell^1$ is complete under $\|\cdot\|_{C^*}$. Now, if $\rho$ is a representation of $\ell^1$ then $\rho(f)$ is an element of some $C^*$ Algebra. Hence, fix a representation $\rho$
$$\|\rho(f \ast g)\| = \|\rho(f) \rho(g)\| \leq \|\rho(f)\| \cdot \|\rho(g)\|$$
$$\leq \sup\{\|\pi(f)\| \cdot \|\pi(g)\| : \pi \text{ is a } * \text{- representation}\}$$
$$= \sup\{\|\pi(f)\| : \pi \text{ is a } * \text{- representation}\} \cdot \sup\{\|\pi(g)\| : \pi \text{ is a } * \text{- representation}\}$$
$$= \|f\|_{C^*} \cdot \|g\|_{C^*}.$$
This implies that \( \|\rho(f \ast g)\| \leq \|f\|_{C^*} \cdot \|g\|_{C^*} \), for an arbitrary representation \( \rho \). Thus,
\[
\|f \ast g\|_{C^*} = \sup\{\|\pi(f \ast g)\| : \pi \text{ is a } * - \text{representation}\} \\
\leq \|f\|_{C^*} \cdot \|g\|_{C^*}
\]

**C*-condition:** Let \( \rho \) be a representation, then
\[
\|\rho(f^* \ast f)\| = \|\rho(f)^* \rho(f)\| = \|\rho(f)\|^2
\]
Thus, since \( \rho \) was arbitrary, \( \|f^* \ast f\|_{C^*} = \|f\|_{C^*}^2 \).

Next, we apply a similar construction with a non-commutative multiplication.

Since our goal is to characterize the Quantum Tori, we look at \( \ell^1(\mathbb{Z}^d) \) with a non-commutative multiplication to develop intuition for showing that the construction of \( \mathcal{A}_\theta \) is universal. We further simplify this construction by only considering \( \ell^1(\mathbb{Z}^2) \) with \( \theta(k, t) = \theta(k, t) = \theta \cdot (k_1 t_2 - k_2 t_1) \) where \( k = (k_1, k_2), t = (t_1, t_2) \in \mathbb{Z}^2 \) and \( \theta \in (0, 1) \). If \( \theta = 0 \) then the Twisted Convolution would be the regular convolution. Also, for the remainder of this section we will denote \( \ell^1(\mathbb{Z}^2) \) as \( \ell^1 \). Notice that the proofs will be similar to those in section 1. We will construct a similar norm by first showing the existence of some representation, which is where the difficulty of this section will lie. By non-commutativity, we could not have a 1-dimensional representation as we did in section 1.

**Definition 6.1.3.2.** If \( f, g \in \ell^1 \), then the **twisted convolution** of \( f, g \) is defined as
\[
(f \ast_{\theta} g)(k) = \sum_{n \in \mathbb{Z}^d} f(n) g(k - n) e^{\pi i \theta(k, n)}.
\]

**Proposition 6.1.3.1.** Let \( f, g \in \ell^1 \), then \( (f \ast_{\theta} g) \in \ell^1 \) with
\[
||f \ast_{\theta} g||_1 \leq ||f||_1 \cdot ||g||_1
\]

**Proof.** This is the same argument as **Proposition 1** since \( |e^{\pi i \theta(k, n)}| = 1 \).

**Theorem 6.1.3.2.** \((\ell^1, +, \ast_{\theta}, \cdot)\) is a unital Banach Algebra.

**Proof.** Again, like **Proposition 2** of this section, the proof of **Theorem 4** of this section is similar to the proof of **Theorem 1** of section 1 except the commutativity. The similarity is due to \( \theta(k, n) \), bilinear and skew-symmetric, and will preserve the substitutions made in the arguments in the proof of **Theorem 1** of section 1. So, with the loss
of commutativity (we will soon show explicitly why $\ast_\theta$ is not commutative), it remains to show that $\delta_0$ is still a left and right unit.

Let $f \in \ell^1$,

$$(f \ast_\theta \delta_0)(k) = \sum_{n \in \mathbb{Z}^d} f(n)\delta_0(k-n)e^{n\pi i \theta(k,n)} = f(k)e^{\pi i \theta(k,k)} = f(k)e^0 = f(k)$$

And,

$$(\delta_0 \ast_\theta f)(k) = \sum_{n \in \mathbb{Z}^d} \delta_0(n)f(k-n)e^{n\pi i \theta(k,n)} = f(k)e^{\pi i \theta(k,0)} = f(k)e^0 = f(k)$$

Thus, $\delta_0$ is a left and right unit. \hfill \square

6.1.4. Involution and $\delta_k$.

**Definition 6.1.4.1.** Let $f \in \ell^1$ then we define the involution of $f$ point-wise as $f^*(k) = f(-k)$.

Again, this is clearly an involution and is continuous since translation and conjugation are continuous. Also, we see in part 2 of Lemma 2, as long as $j \neq m \in \mathbb{Z}^2$ and $m \neq -j$, $\delta_j$ does not commute with $\delta_m$ under $\ast_\theta$.

**Lemma 6.1.4.1.**

1. $\forall m, j \in \mathbb{Z}^2 \delta_j \ast_\theta \delta_m = e^{\pi i \theta(m,j)}\delta_{j+m}$
2. $\forall m, j \in \mathbb{Z}^2 \delta_j \ast_\theta \delta_m = e^{2\pi i \theta(m,j)}\delta_m \ast_\theta \delta_j$
3. $\forall j \in \mathbb{Z}^2 \delta_j$ is unitary with respect to the involution, $\ast$, and twisted convolution, $\ast_\theta$.

**Proof.**

1. **Fix** $m, j \in \mathbb{Z}^2$, then

$$(\delta_j \ast_\theta \delta_m)(j+m) = \sum_{n \in \mathbb{Z}^2} \delta_j(n)\delta_m(j+m-n)e^{n\pi i \theta(j+m,n)}$$

$$= \delta_j(j)\delta_m(j+m-j)e^{\pi i \theta(j+m,j)}$$

$$= \delta_j(j)\delta_m(m)e^{\pi i (\theta(j,j)+\theta(m,j))} = \delta_j(j)\delta_m(m)e^{\pi i (0+\theta(m,j))} = e^{\pi i \theta(m,j)}$$

But, $(\delta_j \ast_\theta \delta_m)(k) = 0$ otherwise. Hence, $\delta_j \ast_\theta \delta_m = e^{\pi i \theta(m,j)}\delta_{j+m}$.

2. **Fix** $m, j \in \mathbb{Z}^2$, then by part 1,

$$\delta_j \ast_\theta \delta_m = e^{\pi i \theta(m,j)}\delta_{j+m} \quad (\Delta)$$

But, $\delta_{j+m} = \delta_{m+j}$. Hence, again by part 1,

$$\delta_m \ast_\theta \delta_j = e^{\pi i \theta(j,m)}\delta_{m+j} = e^{\pi i \theta(j,m)}\delta_{j+m}$$
Now, $\delta_{j+m} = e^{-\pi i \theta(j,m)} \delta_m \ast_\theta \delta_j = e^{\pi i \theta(m,j)} \delta_m \ast_\theta \delta_j$. Therefore, from $(\Delta)$,

$$\delta_j \ast_\theta \delta_m = e^{\pi i \theta(m,j)} \delta_j$$
$$= e^{\pi i \theta(m,j)} \delta_m \ast_\theta \delta_j$$

\[= e^{2\pi i \theta(m,j)} \delta_m \ast_\theta \delta_j\]

3. Let $j \in \mathbb{Z}^2$,

$$\delta_j^* \ast_\theta \delta_j = \delta_{-j} \ast_\theta \delta_j$$
$$= e^{\pi i \theta(j,-j)} \delta_{-j} \ast_\theta \delta_j \text{ by part 1,}$$
$$= e^0 \delta_0 = \delta_0 \text{ since } \theta(j,-j) = 0$$

$$\delta_j^* \ast_\theta \delta_j = \delta_{-j} \ast_\theta \delta_j$$
$$= e^{2\pi i \theta(j,-j)} \delta_j \ast_\theta \delta_{-j} \text{ by part 2,}$$
$$= e^0 \delta_j \ast_\theta \delta_j = \delta_j \ast_\theta \delta_j$$

Therefore, $\delta_j \ast_\theta \delta_j^* = \delta_j \ast_\theta \delta_j = \delta_0$, and so $\delta_j$ is unitary.

Similarly, as in section 1 and by part 1 of Lemma 2, the subalgebra, $\mathcal{C}$, generated by $\{\delta_{e_1}, \delta_{e_2}\}$ is dense in $\ell^1$, where $e_1 = (1,0), e_2 = (0,1)$. Also, we have by part 2 of Lemma 2 that

$$\delta_{e_2} \ast_\theta \delta_{e_1} = e^{2\pi i \theta(e_1,e_2)} \delta_{e_1} \ast_\theta \delta_{e_2} = e^{2\pi i \theta} \delta_{e_1} \ast_\theta \delta_{e_2}.$$ 

Also, the same example in section 1 shows us that $\ell^1$ is not a $C^*$ algebra under the norm $\| \cdot \|_1$ with the above involution and Twisted Convolution.

6.1.5. A Representation. The representation we will construct will be defined on $\mathcal{C}$ (defined above), and will be extended to all of $\ell^1$ by continuity. So, first, we prove a lemma to simplify the multiplication of elements in $\mathcal{C}$.

**Lemma 6.1.5.1.** If $f, g \in \mathcal{C}$, where $f, g$ are the finite sums,

$$f = \sum_{k,l} a_{kl} \delta_{e_2}^k \delta_{e_1}^l \quad \text{and} \quad g = \sum_{m,n} b_{mn} \delta_{e_2}^m \delta_{e_1}^n,$$

where products and powers are with respect to $\ast_\theta$ and $k, l, m, n \in \mathbb{Z}$, then

$$f \ast_\theta g = \sum_{k,l,m,n} a_{kl} b_{mn} e^{-2\pi i lm \theta} \delta_{e_2}^k \delta_{e_1}^l.$$
Proof. Fix \( k, n \in \mathbb{N} \). We show that for all \( l, m \in \mathbb{N} \)

\[
(\delta_{e_2}^k \delta_{e_1}^l)(\delta_{e_2}^m \delta_{e_1}^n) = e^{-2\pi ilm\theta} \delta_{e_2}^{k+m} \delta_{e_1}^{l+n}
\]

But, notice by Lemma 2, \( \delta_{e_1}^{-1} = \delta_{-e_1} \) and \( \delta_{e_2}^{-1} = \delta_{-e_2} \). Thus, the above is equivalent to showing

\[
\delta_{e_1}^l \delta_{e_2}^m = e^{-2\pi ilm\theta} \delta_{e_2}^{m+l} \delta_{e_1}^l
\]

Recall that by the discussion at the end of 6.1.4, \( \delta_{e_2} \delta_{e_1} = e^{2\pi i\theta} \delta_{e_1} \delta_{e_2} \), which is equivalent to \( \delta_{e_1} \delta_{e_2} = e^{2\pi i\theta} \delta_{e_2} \delta_{e_1} \) \((\Delta)\).

Also, we have that, \( \delta_{e_1} = e^{-2\pi i\theta} \delta_{e_2} \delta_{e_1} \delta_{-e_2} \). We proceed by induction on \( l, m \).

**Base Case** \( l=m=1 \):

\[
(\delta_{e_2}^1 \delta_{e_1}^1)(\delta_{e_2}^m \delta_{e_1}^n) = \delta_{e_2}^1 (\delta_{e_1}^1 \delta_{e_2}^m) \delta_{e_1}^n
\]

\[
= \delta_{e_2}^1 (e^{-2\pi i\theta} \delta_{e_2} \delta_{e_1}) \delta_{e_1}^n \quad \text{by } (\Delta)
\]

\[
= e^{-2\pi i\theta} \delta_{e_2}^{k+1} \delta_{e_1}^{l+n} = e^{-2\pi ilm\theta} \delta_{e_2}^{m+l} \delta_{e_1}^l
\]

**Induction Hypothesis:** Assume true for all \( l, m \). Thus,

\[
\delta_{e_1}^l \delta_{e_2}^m = e^{-2\pi ilm\theta} \delta_{e_2}^{m+l} \delta_{e_1}^l
\]

**Induction Step 1:** First, We show that it is true for \( l+1, m \).

\[
\delta_{e_1}^{l+1} \delta_{e_2}^m = \delta_{e_1}^l (\delta_{e_2}^m) \delta_{e_2}^{l+1}
\]

\[
= e^{-2\pi i\theta} \delta_{e_1}^l (\delta_{e_2}^m \delta_{e_2} \delta_{-e_2}) \delta_{e_2}^{l+1} \quad \text{by } (\Delta)
\]

\[
= e^{-2\pi i\theta} \delta_{e_1}^l \delta_{e_2}^m \delta_{e_2}^{l+1} \quad \text{by continuing in the obvious way}
\]

\[
= e^{-2\pi ilm\theta} (e^{-2\pi ilm\theta} \delta_{e_2}^{m+l} \delta_{e_1}^l) \delta_{e_1}^l \quad \text{by Induction Hypothesis}
\]

\[
= e^{-2\pi ilm\theta} \delta_{e_2}^{m+l} \delta_{e_1}^{l+1}
\]

**Induction Step 2:** Next, We show that it is true for \( l, m+1 \).
\[ \delta_{e_1}^{l} \delta_{e_2}^{m+1} = (\delta_{e_1}^{l} \delta_{e_2}^{m}) \delta_{e_2}^{l} \]
\[ = (e^{-2\pi i l m \theta} \delta_{e_2}^{m}) \delta_{e_2}^{l} \]
\[ = e^{-2\pi i l m \theta} \delta_{e_2}^{m} (\delta_{e_1}^{l} \delta_{e_2}^{l}) \delta_{e_2}^{l-1} \delta_{e_2}^{l} \]
\[ = e^{-2\pi i l m \theta} \delta_{e_2}^{m} \delta_{e_1}^{l} \delta_{e_2}^{l-1} \delta_{e_2}^{l} \]
\[ = e^{-2\pi i l m \theta - 2\pi i \theta} \delta_{e_2}^{m+1} \delta_{e_1}^{l} \delta_{e_2}^{l-2} \delta_{e_2}^{l} \]
\[ = e^{-2\pi i l m \theta - 2\pi i \theta} \delta_{e_2}^{m+1} \delta_{e_1}^{l} \delta_{e_2}^{l-2} \delta_{e_2}^{l} \]
\[ = e^{-2\pi i l m \theta - (2+2\pi \theta) \delta_{e_2}^{m+1}} \delta_{e_1}^{l} \delta_{e_2}^{l-2} \delta_{e_2}^{l} \]
\[ = e^{-2\pi i l m \theta - 2\pi \theta} \delta_{e_2}^{m+1} \delta_{e_1}^{l} \delta_{e_2}^{l-2} \delta_{e_2}^{l} \]
\[ = e^{-2\pi i l (m+1) \theta} \delta_{e_2}^{m+1} \delta_{e_1}^{l} \delta_{e_2}^{l} \]

Therefore, by the above comments, \( \forall k, l, m, n \in \mathbb{N} \),
\[ (\delta_{e_2}^{l} \delta_{e_1}^{m})(\delta_{e_2}^{m} \delta_{e_1}^{n}) = e^{-2\pi i l m \theta} \delta_{e_2}^{k+m} \delta_{e_1}^{l+n} \] 

Next, we show that it is in fact true for all \( k, l, m, n \in \mathbb{Z} \). Again, we notice that we only need to show for \( l, m \in \mathbb{Z} \), that
\[ \delta_{e_1}^{l} \delta_{e_2}^{m} = e^{-2\pi i l m \theta} \delta_{e_2}^{m} \delta_{e_1}^{l} . \]

**Case 1:** \( l, m \in \mathbb{Z}^{-} \)

Now, fix \( l, m \in \mathbb{N} \). Then,
\[ \delta_{e_2}^{-m} \delta_{e_1}^{-l} = (\delta_{e_2}^{l} \delta_{e_1}^{m})^{-1} = (e^{-2\pi i l m \theta} \delta_{e_2}^{m} \delta_{e_1}^{l})^{-1} \]
\[ = e^{2\pi i (-l) \theta} \delta_{e_1}^{-l} \delta_{e_2}^{-m} \]

Thus,
\[ \delta_{e_1}^{-l} \delta_{e_2}^{-m} = e^{-2\pi i (-l) \theta} \delta_{e_2}^{-m} \delta_{e_1}^{-l} \]

**Case 2:** \( l \in \mathbb{N}, m \in \mathbb{Z}^{-} \) or \( m \in \mathbb{N}, l \in \mathbb{Z}^{-} \)

We only prove the first part of **Case 2** since the second will come from the trick used in **Case 1**.

Let \( l, m \in \mathbb{N} \) from (\( \Delta \)) a computation (of which I will not show since this proof is already very computationally heavy) shows that
\[ \delta_{e_2}^{-m} = e^{2\pi i l m \theta} \delta_{e_1}^{-l} \delta_{e_2}^{-m} \delta_{e_1} \] (\( \Delta' \))
Therefore,
\[
\delta_{e_1}^l \delta_{e_2}^{-m} = \delta_{e_1}^l (e^{2\pi i m \theta} \delta_{e_2}^{-m} \delta_{e_1}) \quad \text{by (\Delta')}
\]
\[
= e^{2\pi i m \theta} \delta_{e_1}^{-1} \delta_{e_2}^{-m} \delta_{e_1} \quad \text{by (\Delta')}
\]
\[
= e^{4\pi i m \theta} \delta_{e_1}^{-2} \delta_{e_2}^{-m} \delta_{e_1}^2
\]
\[
\vdots
\]
\[
= e^{2\pi i m \theta} \delta_{e_2}^{-m} \delta_{e_1}^l \quad \text{by continuing in the obvious way}
\]
\[
= e^{-2\pi i l (-m) \theta} \delta_{e_2}^{-m} \delta_{e_1}^l
\]
Therefore, by the above comments, \(\forall k,l,m,n \in \mathbb{Z}\),
\[
(\delta_{e_1}^k \delta_{e_2}^l) (\delta_{e_1}^m \delta_{e_2}^n) = e^{-2\pi i m \theta} \delta_{e_2}^{k+m} \delta_{e_1}^{l+n} \quad (\nabla')
\]
Hence,
\[
f \ast \theta g = \left( \sum_{k,l} a_{kl} \delta_{e_2}^k \delta_{e_1}^l \right) \left( \sum_{m,n} b_{mn} \delta_{e_2}^m \delta_{e_1}^n \right)
\]
\[
= \sum_{k,l,m,n} a_{kl} b_{mn} e^{-2\pi i k \theta} \delta_{e_2}^k \delta_{e_1}^l \delta_{e_2}^m \delta_{e_1}^n
\]
\[
= \sum_{k,l,m,n} a_{kl} b_{mn} e^{-2\pi i l (-m) \theta} \delta_{e_2}^{k+m} \delta_{e_1}^{l+n}
\]
As mentioned earlier, we will define a representation on \(C\), and, this representation must in some way unravel the identity in **Lemma 3**.

**Lemma 6.1.5.2.** If \(U, V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})\) defined by \(U = M_{e^{-2\pi it}}\) and \(V f(k) = f(k - \theta)\), then \(U, V \in \mathcal{B}(\ell^2(\mathbb{Z}))\) such that \(U, V\) are unitary with \(U V = e^{2\pi i \theta} V U\).

**Proof.** It is clear that \(U, V \in \mathcal{B}(\ell^2(\mathbb{Z}))\).

**U unitary:** Let \(f, g \in \ell^2(\mathbb{Z})\), then
\[
(U f, g) = \sum_{k \in \mathbb{Z}} e^{2\pi i k} f(k) \overline{g(k)} = \sum_{k \in \mathbb{Z}} f(k) \overline{e^{-2\pi i k} g(k)} = \langle f, U^* g \rangle
\]
Where \(U^* = M_{e^{-2\pi it}}\). And, \(UU^* f(k) = U e^{-2\pi i k} f(k) = e^{2\pi i k} e^{-2\pi i k} f(k) = f(k) = U^* U f(k)\). Thus, \(UU^* = I = U^* U\).

**V unitary:**
\[ \langle Vf, g \rangle = \sum_{k \in \mathbb{Z}} f(k - \theta) \overline{g(k)} \quad \text{let} \quad m = k - \theta \]
\[ = \sum_{m \in \mathbb{Z}} f(m) \overline{g(m + \theta)} = \langle f, V^* g \rangle \]

Where, \( V^* f(k) = f(k + \theta) \). And, \( VV^* f(k) = Vf(k + \theta) = f(k + \theta - \theta) = f(k) = V^* V f(k) \). Thus, \( VV^* = I = V^* V \).

Now,
\[ VU f(t) = (U f)(t - \theta) = e^{2\pi i (t - \theta)} f(t - \theta) = e^{-2\pi i \theta} e^{2\pi i t} f(t) = e^{-2\pi i \theta} UV f(t) \]

Thus, \( UV = e^{2\pi i \theta} VU \).

We may now rephrase Lemma 3, in the following way.

**Lemma 6.1.5.3.** If \( f, g \in B(\ell^2(\mathbb{Z})) \) such that \( f, g \) are the finite sums,
\[ f = \sum_{k,l} a_{kl} U^k V^l \quad \text{and} \quad g = \sum_{m,n} b_{mn} U^m V^n, \]
for \( k, l, m, n \in \mathbb{Z} \), then
\[ fg = \sum_{k,l,m,n} a_{kl} b_{mn} e^{-2\pi i m \theta} U^{k+m} V^{l+n} \]

**Proof.** We only need to make the connection that \( U, V \) share the same properties of \( \delta_e^2, \delta_e^1, \delta_e^{-2}, \delta_e^{-1} \) respectively, which allowed us to prove Lemma 3, that \( U, V \) are unitary and \( UV = e^{2\pi i \theta} VU \).

**Theorem 6.1.5.1.** There exists a \( *_{\theta} \)-representation of \( \ell^1 \)

**Proof.** First, we make the following identification:
Now, define \( \rho : \mathcal{C} \to B(\ell^2(\mathbb{Z})) \) by
\[ \rho(f) = \sum_{k,l} a_{kl} U^k V^l \]
where \( f \) is the finite sum \( f = \sum_{k,l} a_{kl} \delta_{e_2}^k \delta_{e_1}^l \), \( k, l \in \mathbb{Z} \). Now, to show that \( \rho \) is a representation, we technically need to show for any finite sums of powers of \( \delta_{e_2}, \delta_{e_1}, \delta_{e_2}^*, \delta_{e_1}^* \). But, by unitary \( \delta_{e_2}^* = \delta_{e_2}^{-1} \) and \( \delta_{e_1}^* = \delta_{e_1}^{-1} \). Thus, it is enough to show for finite sums of negative and positive powers of just \( \delta_{e_2}, \delta_{e_1} \). It is clear that \( \rho \) is linear. Also, if \( f \) is as above, then \( \|\rho(f)\| = \|\sum_{k,l} a_{kl} U^k V^l\| \leq \sum_{k,l} \|a_{kl} U^k V^l\| = \sum_{k,l} |a_{kl}| = \|f\|_1 \) since \( U, V \) are unitary. But, since \( \mathcal{C} \) is a dense subalgebra of \( \ell^1 \), \( \rho \) is bounded linear map on \( \ell^1 \).
Also note that, $\delta^*_e \mapsto U^*$ and $\delta^*_e \mapsto V^*$. Thus, $\rho(f^*) = \rho(f)^*$.

**Multiplicative:** Let $f, g$ be finite sums as defined in **Lemma 3**, then by **Lemma 3**, 

$$\rho(f \ast g) = \sum_{k,l,m,n} a_{kl} b_{mn} e^{-2\pi i \lambda \theta} U^{k+m} V^{l+n}$$

But, by **Lemma 5**, 

$$\rho(f) \rho(g) = \left( \sum_{k,l} a_{kl} U^k V^l \right) \left( \sum_{m,n} b_{mn} U^m V^n \right) = \sum_{k,l,m,n} a_{kl} b_{mn} e^{-2\pi i \lambda \theta} U^{k+m} V^{l+n}$$

And, so $\rho(f \ast g) = \rho(f) \rho(g)$. Thus, $\rho$ is a $\ast$-representation of $\ell^1$. And, by continuity, this extends to all of $\ell^1$. □

**Proposition 6.1.5.1.** $\rho$ is faithful.

**Proof.** This utilizes a fact about integrated representations. One reference for this is Wiliams’ *Crossed Products of C*-algebras.* □

Therefore, just as we did for the regular convolution, we may define the norm by 

$$\|f\|_{C^*} = \sup \{ \| \pi(f) \| : \pi \text{ is a } \ast \theta \text{- representation} \}.$$ 

The norm properties, including the $C^*$ condition, follow the same proof as section 1, and we still have positive definiteness since $\rho$ is a faithful representation. Hence, once we complete $\ell^1(\mathbb{Z}^2)$ under $\| \cdot \|_{C^*}$, we have that 

$$A_\theta := (\ell^1(\mathbb{Z}^2), +, \ast, \ast, \| \cdot \|_{C^*})$$

is a $C^*$-algebra generated by two unitaries $\delta^*_e, \delta^*_e$ such that $\delta^*_e \delta^*_e = e^{2\pi i \theta} \delta^*_e \delta^*_e$. (♠)

7. **Property (♠) is Universal for Irrational $\theta$**

We accomplish this task by showing that any arbitrary $C^*$-algebra generated by two unitaries satisfying (♠) is isomorphic to $A_\theta$. Now, by the prequels to this section, we have that $A_\theta$ is the completion of the $\| \cdot \|_{C^*}$-closure of the algebra generated by the unitaries $\delta^*_e, \delta^*_e$ satisfying (♠). For the remainder, we denote the norm on $A_\theta$ by $\| \cdot \|_{A_\theta}$. Also, we denote $U := \delta^*_e$ and $V := \delta^*_e$.

Let $\tilde{U}, \tilde{V}$ be two unitaries such that $\tilde{U}, \tilde{V} \in B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that $\tilde{U}, \tilde{V}$ satisfy (♠) as $\tilde{U} \tilde{V} = e^{2\pi i \theta \tilde{V} \tilde{U}}$ for the same $\theta \in (0, 1), \theta \notin \mathbb{Q}$ fixed in section 2.
Define $\varphi : A_\theta \to C^*(\tilde{U}, \tilde{V})$ by $\varphi(U) = \tilde{U}$ and $\varphi(V) = \tilde{V}$ just as we did for the representation in section 2.1.2.

**Theorem 7.0.5.2.** $\varphi$ is an endomorphism.

**Proof.** Let $f \in C$, where $f = \sum_{k,l} a_{kl} U^k V^l$, $k, l \in \mathbb{Z}$ a finite sum. Just as mentioned in the proof of **Theorem 5** of Section 2.1.2, we only need to consider finite sums of positive and negative powers of $U, V$.

Now, we may generalize **Lemma 5** of section 2 by noticing that $\varphi$ is a representation of $A_\theta$ such that $\varphi(f) = \sum_{k,l} a_{kl} \tilde{U}^k \tilde{V}^l$. Hence,

$$||\varphi(f)||_{B(H)} \leq ||f||_{A_\theta}$$

By, definition of the norm on $A_\theta$. Thus, $\varphi$ is well defined and contractive on $C$ onto $C^*(\tilde{U}, \tilde{V})$, and therefore extends by continuity to a homomorphism of $A_\theta$ onto $C^*(\tilde{U}, \tilde{V})$.

At this point, these notes essentially become a companion to Davidson’s $C^*$ - Algebras By Examples, Section VI.1. Thus, our method for showing that $\varphi$ is in fact an injection follows the method of finding a unique trace, which will be used to show that $A_\theta$ is simple. So, we continue by constructing a trace.

**7.1. Unique Trace.** The above gives that if we have any $C^*$-algebra generated by two unitaries satisfying (♠), $B$, there is an endomorphism from $A_\theta$ to $B$. Next, fix $(\lambda, \mu) \in \mathbb{T}^2$. Then,

$$\lambda U \mu V = \lambda \mu UV$$

$$= \lambda \mu e^{2\pi i \theta} V U$$

$$= e^{2\pi i \theta} \mu V \lambda U$$

Furthermore, $(\lambda U)(\lambda U)^* = \lambda \bar{\lambda} U U^* = U U^* = I = (\lambda U)^*(\lambda U)$. And, similarly $(\mu V)(\mu V)^* = I = (\mu V)^*(\mu V)$. Hence, $(\lambda U, \mu V)$ are two unitaries satisfying (♠). So, there is an endomorphism $\rho_{\lambda, \mu}$ from $A_\theta$ to $C^*(\lambda U, \mu V)$ such that $\rho_{\lambda, \mu}(U) = \lambda U$ and $\rho_{\lambda, \mu}(V) = \mu V$. Now, define $\Gamma : \mathbb{T}^2 \to Aut(A_\theta)$ by $(\lambda, \mu) \mapsto \rho_{\lambda, \mu}$. Before we show that $\Gamma$ is well-defined, we show that itself is a homomorphism.

**Lemma 7.1.0.4.** $\Gamma$ is a homomorphism into $End(A_\theta)$.

**Proof.** Let $(\lambda, \mu), (\alpha, \beta) \in \mathbb{T}^2$.

Then,

$$\Gamma((\lambda, \mu)(\alpha, \beta))(U) = \Gamma((\lambda \alpha, \mu \beta))(U) = \rho_{\lambda \alpha, \mu \beta}(U) = \lambda \alpha U$$

$$= \rho_{\lambda, \mu}(\alpha U) = \rho_{\lambda, \mu}(\rho_{\alpha, \beta}(U)) = \Gamma((\lambda, \mu))(\alpha, \beta)(U)$$
And, similarly, $\Gamma(\lambda, \mu)((\alpha, \beta))(V) = \Gamma((\lambda, \mu))((\alpha, \beta))(V)$. Thus, this can be extended to elements in $\mathcal{C}$. Thus, $\Gamma$ is a homomorphism.

\[\square\]

**Proposition 7.1.0.2.** $\Gamma$ is well-defined.

*Proof.* Fix $((\lambda, \mu)) \in T^2$. Now, by above $\rho_{\lambda, \mu}$ is an endomorphism on $A_{\theta}$. Next, notice that $\rho_{1, 1}$ is the identity map on $A_{\theta}$ and by the above Lemma 6, $\rho_{\lambda, \mu}$ is the left and right inverse for $\rho_{\lambda, \mu}$. Thus, $\rho_{\lambda, \mu}$ is and automorphism, and by $(\lambda, \mu) \in T^2$ arbitrary, $\Gamma$ is well-defined.

\[\square\]

Before we continue, we analyze $A_0$. Although the point has been made that we only consider irrational rotations, this example still illuminates how these maps that construct the trace act. Before we analyze the case of $A_0$, we notice that it is simply $C(T^2)$. This becomes clear when one realizes that $C(T^2)$ is the $C^*$-algebra generated by the unitary operators, the coordinate maps, call them $U', V'$, for which $U'V' = V'U' = e^{2\pi i \cdot 0}V'U'$. Now, $\rho_{\lambda, \mu}(U'(x, y)) = \lambda U'(x, y) = \lambda x$ and $\rho_{\lambda, \mu}(V'(x, y)) = \mu y$.

Next, since every $f \in C(T^2)$ is a limit of polynomials of $U', V'$, $\rho_{\lambda, \mu}(f(x, y)) = f(\lambda x, \mu y)$.

Thus, when $\rho_{\lambda, \mu}$ is applied to $f$, we get a rotation of the the original function in each coordinate.

**Lemma 7.1.0.5.** Fix $A \in A_{\theta}$, then $F_A : T^2 \rightarrow A_{\theta}$ defined by $(\lambda, \mu) \mapsto \rho_{\lambda, \mu}(A)$ is norm continuous.

*Proof.* Let $A \in A_{\theta}$. Let $a \in T^2$, such that $\|a\|_{T^2} = 1$. Now, by Proposition 4, since $\Gamma$ is well-defined, $\rho_a$ is an automorphism and therefore contractive. Hence, since $F_A$ is linear, $\|F_A(a)\|_{A_{\theta}} = \|\rho_a(A)\|_{A_{\theta}} \leq \|A\|_{A_{\theta}} = \|A\|_{A_{\theta}} \|a\|_{T^2}$ implies that $F_A$ is norm continuous since $A$ is fixed.

\[\square\]

So, looking back at $C(T^2)$, Lemma 7 tells us that the function, $F_f$ whose image is every coordinate rotation of $f$ is norm contintuous.

Also, Lemma 7 allows us to define the following two functions from $A_{\theta}$ to $A_{\theta}$.

$$E_1(A) = \int_0^1 F_A(1, e^{2\pi it}) \ dt$$
and

\[ E_2(A) = \int_0^1 F_A(e^{2\pi i t}, 1) \, dt \]

Again, before we move on to prove some nice properties, let’s see what \( E_1, E_2 \) do when applied to continuous functions on \( \mathbb{T}^2 \).

\[ E_1(U') = \int_0^1 F_{U'}(1, e^{2\pi i t}) \, dt \]
\[ = \int_0^1 \rho_{1,e^{2\pi i t}}(U') \, dt = \int_0^1 1 \cdot U' \, dt \]
\[ = U' \]

And,

\[ E_1(V') = \int_0^1 F_{V'}(1, e^{2\pi i t}) \, dt \]
\[ = \int_0^1 \rho_{1,e^{2\pi i t}}(V') \, dt = \int_0^1 e^{2\pi i t} \cdot V' \, dt \]
\[ = 0 \]

Now, if we consider a finite sum, \( \sum_{k,l} a_{kl} U_k V_l \), then we will show shortly that

\[ E_1 \left( \sum_{k,l} a_{kl} U_k V_l \right) = \sum_k a_{k0} U_k. \]

Now, for an arbitrary \( f \in C(\mathbb{T}^2) \) the \( a_{kl} \) are analogous to the Fourier Coefficients of \( f \). Note that \( f \in L^1(\mathbb{T}^2, \mu) \), where \( \mu \) is the Haar measure. By the Fourier Inversion Theorem, there exists a unique Haar measure \( \nu \) on the Pontryagin Dual of \( \mathbb{T}^2, \mathbb{Z}^2 \), which is \( \mathbb{Z}^2 \). So, we denote an element of \( \mathbb{Z}^2 \) as \( (z_k, z_l) \) to identify it as both a homomorphism into the circle and an element of \( \mathbb{Z}^2 \). And, further note that, since \((0,0)\) is the identity in \((\mathbb{Z}^2, +)\), \((0,0)\) is identified as the identity in \( \mathbb{T}^2 \), which is the homomorphism such that for all \((x,y) \in \mathbb{T}^2, (x,y) \mapsto 1 \). Therefore,

\[ f((x,y)) = \int_{\mathbb{Z}^2} \hat{f}(z_k, z_l) \, (z_k, z_l)(x,y) \, d\nu(z_k, z_l) \]

where \( \hat{f} \) is the Fourier Transform

\[ \hat{f}((z_k, z_l)) = \int_{\mathbb{T}^2} f(x,y) \, (z_k, z_l)(x,y) \, d\mu(x,y). \]

Thus,

\[ E_1(f(x,y)) = \int_{\mathbb{Z}^2} \hat{f}(z_k, 0) \, (z_k, 0)(x,y) \, d\nu(z_k, 0) \]

And, similarly,
\[ E_2(f(x, y)) = \int_{\mathbb{Z}^2} \hat{f}(0, z_1) (0, z_1)(x, y) \, d\nu(0, z_1) \]

Notice the following (again this will be proven later formally in general),

\[ E_2(E_1(f(x, y))) = E_1\left( \int_{\mathbb{Z}^2} \hat{f}(z_k, 0) (z_k, 0)(x, y) \, d\nu(z_k, 0) \right) \]
\[ = \int_{\mathbb{Z}^2} \hat{f}(0, 0) (0, 0)(x, y) \, d\nu(0, 0) \]
\[ = \int_{\mathbb{Z}^2} \hat{f}(0, 0) \cdot 1 \, d\nu(0, 0) \]
\[ = \hat{f}(0, 0) \]
\[ = \int_{\mathbb{T}^2} f(r, s) (0, 0)(r, s) \, d\mu(r, s) \]
\[ = \int_{\mathbb{T}^2} f(r, s) \, d\mu(r, s) \quad \forall (x, y) \in \mathbb{T}^2 \]

Thus,

\[ E_2(E_1(f)) = \int_{\mathbb{T}^2} f(r, s) \, d\mu(r, s) \in \mathbb{C} \]

It can also be shown that \( E_2(E_1(f)) = E_1(E_2(f)) = \tau(f) \)

And, if \( g \in C(\mathbb{T}^2) \),

\[ \tau(fg) = \int_{\mathbb{T}^2} f(r, s)g(r, s) \, d\mu(r, s) = \int_{\mathbb{T}^2} g(r, s)f(r, s) \, d\mu(r, s) = \tau(gf) \]

So, in fact it seems that based on our construction in \( C(\mathbb{T}^2) \), the way we defined \( \tau \) might be the natural way to define a trace on \( \mathcal{A}_\theta \).

We will prove the following lemmas for \( E_1 \) and notice that the arguments are symmetric for \( E_2 \)

**Lemma 7.1.0.6.** \( E_1 \) is unital positive contractive and faithful.

**Proof.** **Unital:** Recall that \( \mathcal{A}_\theta \) has unit \( I = \delta_0 \). Since \( \rho_{\lambda, \mu} \) is an automorphism for all \((\lambda, \mu) \in \mathbb{T}^2\), \( \rho_{1,e^{2\pi it}} \) is an automorphism for all \( t \in [0, 1] \). Thus, \( \rho_{1,e^{2\pi it}}(I) = I \). Hence,

\[ E_1(I) = \int_0^1 \rho_{1,e^{2\pi it}}(I) \, dt = 1 \cdot I = I \]

**Contractive:** Now, any partition of \([0, 1]\), \( P = \{x_0, \ldots, x_n\} \) with \( \Delta x_j = x_j - x_{j-1} > 0 \) satisfies \( \sum_{j=1}^n \Delta x_j = 1 \). Also, given an \( A \in \mathcal{A}_\theta \) and a convex combination \( \sum_{j=1}^n a_j \rho_{1,e^{2\pi i\alpha_j}}(A) \),
\[
\left\| \sum_{j=1}^{n} a_j \rho_{1,e^{2\pi i a_j}} (A) \right\|_{\mathcal{A}_\theta} \leq \sum_{j=1}^{n} |a_j| \left\| \rho_{1,e^{2\pi i a_j}} (A) \right\|_{\mathcal{A}_\theta} \\
\leq \sum_{j=1}^{n} |a_j| \left\| A \right\|_{\mathcal{A}_\theta} \quad \text{by } \rho_{1,e^{2\pi i a_j}}, \text{ automorphism} \\
\leq \left\| A \right\|_{\mathcal{A}_\theta} \quad \text{by convex combination.}
\]

Thus, by the discussion of the arbitrary partition of \([0,1]\) above and the definition of the Riemann Integral, \(\left\| E_1 (A) \right\|_{\mathcal{A}_\theta} \leq \left\| A \right\|_{\mathcal{A}_\theta}\). And, since \(A\) was arbitrary, \(\left\| E_1 \right\| \leq 1\). Furthermore, by \(E_1\) unital, \(\left\| E_1 \right\| = 1\).

**Positive and Faithful:** Assume that \(A > 0\). Since for all \(t \in [0,1]\) \(\rho_{1,e^{2\pi it}}\) is an automorphism, it is both faithful and positive. Thus, \(\rho_{1,e^{2\pi it}}(A) > 0\). Also, since the Riemann Integral over \([0,1]\) is a convex combination of \(\rho_{1,e^{2\pi it}}(A)\) over \([0,1]\) with positive scalars, \(E_1(A) > 0\). Thus, \(E_1\) is both faithful and positive.

\[\square\]

**Lemma 7.1.0.7.** For all \(f, g \in C(\mathbb{T})\) and \(A \in \mathcal{A}_\theta\),
\[
E_1(f(U)A g(U)) = f(U)E_1(A)g(U).
\]
Furthermore, for finite sums \(\sum_{k,l} a_{kl} U^{k}V^{l}\),
\[
E_1(\sum_{k,l} a_{kl} U^{k}V^{l}) = \sum_{k} a_{k0} U^{k},
\]
and \(E_1\) is idempotent and maps \(\mathcal{A}_\theta\) onto \(C^*(U)\).

**Proof.** Now, for all \(t \in [0,1]\), \(\rho_{1,e^{2\pi it}}(U) = U\), and so \(\rho_{1,e^{2\pi it}}\) is the identity on \(C^*(U)\). But, since \(U\) is unitary, the spectrum of \(U\), \(\sigma(U)\) is the unit circle. Hence, \(C^*(U)\) is isomorphic to \(C(\sigma(U)) = C(\mathbb{T})\). Let \(f \in C(\mathbb{T})\), then \(f(U) \in C^*(U)\), and \(\rho_{1,e^{2\pi it}}(f(U)) = f(U)\). Therefore, since \(\rho_{1,e^{2\pi it}}\) is a homomorphism, for \(f, g \in C(\mathbb{T})\),
\[
E_1(f(U)A g(U)) = \int_{0}^{1} \rho_{1,e^{2\pi it}}(f(U)A g(U)) \ dt \\
= \int_{0}^{1} \rho_{1,e^{2\pi it}}(f(U)) \rho_{1,e^{2\pi it}}(A) \rho_{1,e^{2\pi it}}(g(U)) \ dt \\
= \int_{0}^{1} f(U) \rho_{1,e^{2\pi it}}(A) g(U) \ dt \\
= f(U) \left( \int_{0}^{1} \rho_{1,e^{2\pi it}}(A) \ dt \right) g(U) = f(U)E_1(A)g(U)
\]
Thus, since for any \(k \in \mathbb{Z}\), \(f_k(x) = x^k\) is continuous on \(\mathbb{T}\),
\[ E_1(U^kV^l) = E_1(f_k(U)V^l) = f_k(U)E_1(V^l) \]
\[ = U^k \int_0^1 \rho_1 e^{2\pi ilt}(V^l) \, dt = U^k \left( \int_0^1 e^{2\pi ilt} \, dt \right) V^l \]

Now, if \( l \neq 0 \), then \( \int_0^1 e^{2\pi ilt} \, dt = 0 \), so \( E_1(U^kV^l) = 0 \). Next, if \( l = 0 \), then \( E_1(U^kV^0) = E_1(U^kI) = U^k \int_0^1 \rho_1 e^{2\pi ilt}(I) \, dt = U^k \int_0^1 I \, dt = U^k \)

Hence, \( E_1(U^kV^l) = \delta_{l0}U^k \), where \( \delta_{l0} \) is the Kronecker Delta. Hence, by linearity of the integral, for a finite sum,
\[ E_1(\sum_{k,l} a_{kl}U^kV^l) = \sum_k a_{k0}U^k, \]
which also implies that \( E_1 \) is the identity on \( C^*(U) \). Thus, \( E_2^1 = E_1 \), and, furthermore by density, the range of \( E_1 \) is exactly \( C^*(U) \).

\( \square \)

In our example of \( C(T^2) \), the above lemma allowed us to cancel the fourier coefficients in one coordinate.

**Lemma 7.1.0.8.** For all \( A \in A_\theta \), then
\[ E_1(A) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j AU^{-j}. \]

**Proof.** Recall that in the start of Section 3., we defined \( U = \delta_{e_2}, V = \delta_{e_1} \). Further recall that in section 2.1.2 , we showed in the proof of **Lemma 3** that for \( k, l, m, n \in \mathbb{Z} \)
\[ U^kV^l \sum_{m} V^n = e^{-2\pi i kl \theta} U^{k+m}V^{l+n} \quad (\Delta) \]

Thus, if \( j, k, l \in \mathbb{Z} \), then
\[ U^j(U^kV^l)U^{-j} = U^j(U^kV^lU^{-j}V^0) \]
\[ = U^j(e^{-2\pi il(-j)\theta} U^{k+(-j)}V^l) \quad \text{by (}\Delta\text{)} \]
\[ = e^{2\pi il\theta}U^j U^{-j} U^kV^l = e^{2\pi il\theta} U^kV^l \]

Thus, \( U^j(U^kV^l)U^{-j} = e^{2\pi il\theta} U^kV^l \quad (\Delta') \).

Next, note that since \( \theta \) is irrational, for any \( l \in \mathbb{Z} \), \( e^{2\pi il\theta} \neq 1 \). Thus, we may compute the following identity, which is only well-defined and non-trivial for all \( l \in \mathbb{Z} \), when \( \theta \) is irrational,
\[ \sum_{j=-n}^{n} e^{2\pi il\theta} = \frac{e^{-2\pi il\theta}(-1 + e^{2\pi i(2n+1)\theta})}{-1 + e^{2\pi i\theta}} \quad (\Delta'') \]
Further, note that the modulus of the right hand side is bounded by 1. Assume $l \neq 0$.

\[
\lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j(U^kV^l)U^{-j} = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} e^{2\pi ilj\theta} U^kV^l \quad \text{by } (\Delta'),
\]

(2)

\[
\text{next by } (\Delta'') \quad = \left[ \lim_{n \to \infty} \frac{1}{2n+1} \left( \frac{e^{-2\pi inl\theta}(-1 + e^{2\pi il(2n+1)\theta})}{-1 + e^{2\pi il\theta}} \right) \right] U^kV^l
\]

\[
= 0 \cdot U^kV^l = 0
\]

We realize another importance of $\theta \notin \mathbb{Q}$ on the right hand side of (1). If $\theta$ were rational then there would be a countable number of $l \in \mathbb{Z}$ (any multiple of the denominator of $\theta$) such that $e^{2\pi il\theta} = 1$ for all $j \in \mathbb{Z}$.

Thus for each of these non-zero $l \in \mathbb{Z}$,

\[
\lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j(U^kV^l)U^{-j} = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} e^{2\pi ilj\theta} U^kV^l
\]

\[
= \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} 1 \cdot U^kV^l
\]

\[
= \left( \lim_{n \to \infty} \frac{1}{2n+1} \cdot 2n \right) U^kV^l = U^kV^l \neq 0
\]

This would ruin the possibility of $E_1(U^kV^l) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j(U^kV^l)U^{-j}$ since $E_1(U^kV^l) = \delta_0 U^k$ by Lemma 9, which we will come to see is crucial in our proof that $A_{\theta}$ is simple. Hopefully, it is now clearer as to why we only consider irrational $\theta$.

Now, if $l = 0$, then

\[
\lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j(U^kV^0)U^{-j} = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} e^{2\pi i(0)j\theta} U^kV^0 \quad \text{by } (\Delta')
\]

\[
= \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^k
\]

\[
= \left( \lim_{n \to \infty} \frac{1}{2n+1} \cdot 2n \right) U^k = U^k
\]

Therefore,

\[
\lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j(U^kV^l)U^{-j} = \delta_0 U^k = E_1(U^kV^l).
\]
Therefore, by linearity and continuity of $E_1$, for all $A \in \mathcal{A}_\theta$, 
\[
E_1(A) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j A U^{-j}. 
\]

All the lemmas for $E_1$ are true for $E_2$ in the obvious way.

**Lemma 7.1.0.9.** $E_1 E_2 = E_2 E_1$ and is scalar valued.

**Proof.** By composition, $E_1 E_2$ and $E_2 E_1$ are continuous and linear.

Fix $U^k V^l$ for some $k, l \in \mathbb{Z}$.

Then,
\[
E_2(E_1(U^k V^l)) = E_2(\delta_{00} U^k) = \delta_{00} E_2(U^k) = \delta_{00} E_2(U^0 V^0) = \delta_{(l-k)0} I 
\]

And, similarly, $E_2(E_1(U^k V^l)) = \delta_{(l-k)0} I$. Hence, they equal on the monomials. Thus, by linearity, they equal on the finite sums and by continuity they agree on all of $\mathcal{A}_\theta$. By the computation above they not only agree, but for any $A \in \mathcal{A}_\theta$, $E_2(E_1(A)) = E_1(E_2(A)) = aI$, where $a \in \mathbb{C}$. So, it is scalar valued.

By **Lemma 11**, let $\tau = E_1 E_2 = E_2 E_1$.

**Theorem 7.1.0.3.** $\tau$ is a faithful unital scalar valued trace on $\mathcal{A}_\theta$.

**Proof.** Again by **Lemma 11**, $\tau$ is scalar valued. Now, since $I = U^0 V^0$, by the computation in **Lemma 11**, $\tau(I) = I$. Thus, $\tau$ is unital. $\tau$ is faithful, positive, and contractive because both $E_1$ and $E_2$ are by **Lemma 8**. Hence, $\|\tau\| = 1$.

Now, recall from the proof of **Lemma 3** in section 2.1.2, that for $k, l, m, n \in \mathbb{Z}$
\[
U^k V^l U^m V^n = e^{-2\pi ilm \theta} U^{k+m} V^{l+n} \quad (\Delta)
\]

Thus, consider the monomials $U^k V^l$ and $U^m V^n$
\[
\tau((U^k V^l)(U^m V^n)) = e^{-2\pi ilm \theta} \tau(U^{k+m} V^{l+n}) \quad \text{by linearity and } \Delta \\
= e^{-2\pi i(lm) \theta} \delta_{(k+m+l+n)0} I \quad \text{by proof of **Lemma 10**}
\]
\[
\tau((U^m V^n)(U^k V^l)) = e^{-2\pi ink \theta} \tau(U^{k+m} V^{l+n}) = e^{-2\pi i(kn) \theta} \delta_{(k+m+l+n)0} I 
\]

Now, assume that $k + m = l + n = 0$, then $kn + mn = 0$ and $lm + nm = 0$. And, so by commutativity and cancellation of integers,
\[ kn = lm. \] Hence, \( k + m = l + n = 0 \) if and only if \( e^{-2\pi i (nk)} = e^{-2\pi i (lm)} \). Therefore, \( \tau((U^k V^l)(U^m V^n)) = \tau((U^m V^n)(U^k V^l)) \). By linearity of \( \tau \), this extends to all finite sums of powers of \( U, V \), and then by continuity \( \tau \) is a trace on all of \( \mathcal{A}_\theta \).

**Theorem 7.1.0.4.** \( \tau \) is the unique trace on \( \mathcal{A}_\theta \).

**Proof.** Assume there exist another trace, \( \rho \), on \( \mathcal{A}_\theta \). Then, let \( A \in \mathcal{A}_\theta \), since \( \rho \) is a trace, \( \rho(A) = \rho((AU^{-j})U^j) = \rho(U^j(AU^{-j})) = \rho(U^jAU^{-j}) \) \((\forall)\), for any \( j \in \mathbb{N} \). Next, by **Lemma 10**, \[
\rho(E_1(A)) = \rho \left( \lim_{n \to \infty} \frac{1}{2n + 1} \sum_{j=-n}^{n} U^jAU^{-j} \right)
= \lim_{n \to \infty} \rho \left( \frac{1}{2n + 1} \sum_{j=-n}^{n} U^jAU^{-j} \right) \text{ by } \rho \text{ continuous}
= \lim_{n \to \infty} \frac{1}{2n + 1} \sum_{j=-n}^{n} \rho(U^jAU^{-j}) \text{ by } \rho \text{ linear}
= \lim_{n \to \infty} \frac{1}{2n + 1} \sum_{j=-n}^{n} \rho(A) \text{ by } (\forall)
= \lim_{n \to \infty} \frac{2n}{2n + 1} \rho(A) = \rho(A).
\]

And, similarly, \( \rho(E_2(A)) = \rho(A) \). Recall that the use of **Lemma 10** required that we only consider irrational \( \theta \), and these last two statements rely heavily on **Lemma 10**. In the coming theorem, we will use the above calculation to prove that \( \mathcal{A}_\theta \) is simple. So, this is a reminder of the importance of choosing \( \theta \) irrational in these arguments.

Now, since \( \rho \) is assumed to be a trace, \( \rho(I) = I \). Also, \( A \) arbitrary and \( E_2(A) \in \mathcal{A}_\theta \) and further \( E_1(E_2(A)) \in \mathcal{A}_\theta \) allows us to make the following calculation along with \( \tau(A) = aI \) for some \( a \in \mathbb{C} \),

\[
\tau(A) = aI = a\rho(I) = \rho(aI) \text{ by linearity}
= \rho(\tau(A)) = \rho(E_1(E_2(A)))
= \rho(E_2(A)) = \rho(A).
\]

Thus, \( \rho(A) = \tau(A) \) for all \( A \in \mathcal{A}_\theta \). Hence, \( \tau \) is the unique trace on the irrational rotation algebra \( \mathcal{A}_\theta \). \(\square\)

**Theorem 7.1.0.5.** \( \mathcal{A}_\theta \) is simple.
Proof. Let \( J \) be a non-zero two-sided ideal of \( A_\theta \). Since \( J \) is non-zero, there exists some \( x \in J \). Now, \( x^* \in A_\theta \), and by left ideal, \( b = x^*x \in J \), where \( b \) is positive. By assumption, \( U_jb^U_j^{-1} \in J \) for all \( j \in \mathbb{Z} \). Also, since \( J \) is a closed subalgebra, it is closed under linear combinations and limits.

Hence,
\[
E_1(b) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U_jb^U_j^{-1} \in J.
\]

Furthermore, by the same reason,
\[
\tau(b) = E_2(E_1(b)) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} V_j E_1(b) V_j^* \in J.
\]

But, \( \tau \) is faithful, so \( \tau(b) \) is a non-zero multiple of the identity. Thus, \( \tau(b) = aI \in J \), where \( 0 \neq a \in \mathbb{C} \). Therefore, \( aI \in J \), which implies that \( a^{-1}I(aI) = I \in J \). Thus, the identity is in \( J \) and so \( J = A_\theta \). Hence, \( A_\theta \) is simple. \( \square \)

**Corollary 7.1.0.1.** If \( \widetilde{U}, \widetilde{V} \) are any two unitary elements satisfying \((\heartsuit)\), then \( A_\theta \) is canonically isomorphic to \( C^*(\widetilde{U}, \widetilde{V}) \).

Proof. By **Theorem 6** of section 3., \( \varphi \) is an endomorphism from \( A_\theta \) onto \( C^*(\widetilde{U}, \widetilde{V}) \). Thus, \( \varphi \) is onto and therefore a non-trivial homomorphism. Thus, since \( A_\theta \) is simple by **Theorem 9**, \( \ker(\varphi) = \{0\} \) since the kernel of a homomorphism is a two-sided ideal. Hence, \( \varphi \) is an injective \(*\)-homomorphism, which implies that it is an isometry. Thus, \( A_\theta \) is canonically isomorphic to \( C^*(\widetilde{U}, \widetilde{V}) \). \( \square \)

Hence, a \( C^* \)-Algebra generated by two unitaries satisfying \((\heartsuit)\) is unique up to isometric \(*\)-isomorphism.

**References**


