6.1

8. Find the inverse Laplace transform of the function \( \frac{5}{3s} \).

\[
L^{-1}\left[ \frac{5}{3s} \right] = \frac{5}{3} L^{-1}\left[ \frac{1}{s} \right] = \frac{5}{3} \cdot 1 = \frac{5}{3}.
\]

10. Find the inverse Laplace transform of the function \( \frac{14}{(3s + 2)(s - 4)} \).

We can rewrite this function as \( \frac{14}{(3s + 2)(s - 4)} = \frac{-3}{3s + 2} + \frac{1}{s - 4} \) using partial fractions. Then:

\[
L^{-1}\left[ \frac{14}{(3s + 2)(s - 4)} \right] = L^{-1}\left[ \frac{-3}{3s + 2} + \frac{1}{s - 4} \right] = -L^{-1}\left[ \frac{-3}{3s + 2} \right] + L^{-1}\left[ \frac{1}{s - 4} \right] = -\frac{1}{e^{2t}} + e^{4t}.
\]

12. Find the inverse Laplace transform of the function \( \frac{5}{(s - 1)(s - 2)} \).

We can rewrite this function as \( \frac{5}{(s - 1)(s - 2)} = \frac{5}{s - 2} - \frac{5}{s - 1} \) using partial fractions. Then:

\[
L^{-1}\left[ \frac{5}{(s - 1)(s - 2)} \right] = L^{-1}\left[ \frac{5}{s - 2} - \frac{5}{s - 1} \right] = 5 L^{-1}\left[ \frac{1}{s - 2} \right] - 5 L^{-1}\left[ \frac{1}{s - 1} \right] = 5(e^{2t} - e^{t}).
\]

14. Find the inverse Laplace transform of the function \( \frac{2s^2 + 3s - 2}{s(s + 1)(s - 2)} \).

We can rewrite this function as \( \frac{2s^2 + 3s - 2}{s(s + 1)(s - 2)} = \frac{1}{s} - \frac{1}{s + 1} + \frac{2}{s - 2} \) using partial fractions. Then:

\[
L^{-1}\left[ \frac{2s^2 + 3s - 2}{s(s + 1)(s - 2)} \right] = L^{-1}\left[ \frac{1}{s} - \frac{1}{s + 1} + \frac{2}{s - 2} \right] = 1 - e^{-t} + 2e^{2t}.
\]
dy \over dt} + 5y = e^{-t}, \quad y(0) = 2

(a) Take the Laplace transform of both sides:

\[ L \left( \frac{dy}{dt} \right) + 5L[y] = L[e^{-t}] \]

\[ sL[y] - y(0) + 5L[y] = L[e^{-t}] \]

(b) Plug in the initial condition and solve for the Laplace transform:

\[ sL[y] - 2 + 5L[y] = L[e^{-t}] \]

\[ (s + 5)L[y] = \frac{1}{s + 1} + 2 \]

\[ L[y] = \frac{2s + 3}{(s + 5)(s + 1)} = \frac{7}{4(s + 5)} + \frac{1}{4(s + 1)} \]

(c) Take the inverse Laplace transform of both sides:

\[ L^{-1} [L[y]] = L^{-1} \left[ \frac{7}{4(s + 5)} + \frac{1}{4(s + 1)} \right] \]

\[ y = \frac{7}{4} e^{-5t} + \frac{1}{4} e^{-t} \]

\[ \frac{dy}{dt} + 4y = 6, \quad y(0) = 0 \]

(a) Take the Laplace transform of both sides:

\[ L \left( \frac{dy}{dt} \right) + 4L[y] = 6L[1] \]

\[ sL[y] - y(0) + 4L[y] = 6L[1] \]

(b) Plug in the initial condition and solve for the Laplace transform:

\[ sL[y] - 0 + 4L[y] = 6L[1] \]

\[ (s + 4)L[y] = \frac{6}{s} \]

\[ L[y] = \frac{6}{s(s + 4)} = \frac{3}{2s} - \frac{3}{2(s + 4)} \]

(c) Take the inverse Laplace transform of both sides:

\[ L^{-1} [L[y]] = L^{-1} \left[ \frac{3}{2s} - \frac{3}{2(s + 4)} \right] \]

\[ y = \frac{3}{2} - \frac{3}{2} e^{-4t} \]

20. Use the following equation: \( \frac{dy}{dt} = -y + 2, \ y(0) = 4 \).

(a) We have \( L \left( \frac{dy}{dt} \right) = -L[y] + 2 \ L[1] \implies s \ L[y] - y(0) = -L[y] + \frac{2}{s} \).

(b) We have:

\[ s \ L[y] - y(0) = -L[y] + \frac{2}{s} \]

\[ \implies s \ L[y] + L[y] = \frac{2}{s} + y(0) \]

\[ \implies L[y] = \frac{2 + 4s}{s(s + 1)} \]
(c) Rewrite \( \frac{2s + 4}{s(s + 1)} \) as \( \frac{2}{s} + \frac{2}{s + 1} \) using partial fractions. Then we have:

\[
L^{-1} \left[ \frac{2}{s} \right] + L^{-1} \left[ \frac{2}{s + 1} \right] = 2 L^{-1} \left[ \frac{1}{s} \right] + 2 L^{-1} \left[ \frac{1}{s + 1} \right]
\]

\[
= 2(e^t + e^{-t})
\]

\[
= 2 + 2e^{-t}.
\]

22. \( \frac{dy}{dt} - 2y = t, \quad y(0) = 0 \)

(a) Take the Laplace transform of both sides:

\[
L \left[ \frac{dy}{dt} \right] - 2L[y] = L[t]
\]

\[
sL[y] - y(0) - 2L[y] = L[t]
\]

(b) Plug in the initial condition and solve for the Laplace transform:

\[
sL[y] - 0 - 2L[y] = \frac{1}{s^2}
\]

\[
L[y] = \frac{1}{s^2(s - 2)} = \frac{-1}{2s^2} - \frac{1}{4s} + \frac{1}{4s(s - 2)}
\]

(c) Take the inverse Laplace transform of both sides:

\[
L^{-1} [L[y]] = L^{-1} \left[ \frac{-1}{2s^2} - \frac{1}{4s} + \frac{1}{4s(s - 2)} \right]
\]

\[
y = \frac{-1}{2}t - \frac{1}{4} + \frac{1}{4}e^{2t}
\]

24. \( \frac{dy}{dt} + 4y = 2 + 3t, \quad y(0) = 1 \)

(a) Take the Laplace transform of both sides:

\[
L \left[ \frac{dy}{dt} \right] + 4L[y] = 2L[1] + 3L[t]
\]

\[
sL[y] - y(0) + 4L[y] = 2L[1] + 3L[t]
\]

(b) Plug in the initial condition and solve for the Laplace transform:

\[
sL[y] - 1 + 4L[y] = \frac{2}{s} + \frac{3}{s^2}
\]

\[
(s + 4)L[y] = \frac{2}{s} + \frac{3}{s^2} + 1
\]

\[
L[y] = \frac{s^2 + 2s + 3}{s^2(s + 4)} = \frac{5}{16s} + \frac{3}{4s^2} + \frac{11}{16(s + 4)}
\]

(c) Take the inverse Laplace transform of both sides:

\[
L^{-1} [L[y]] = L^{-1} \left[ \frac{5}{16s} + \frac{3}{4s^2} + \frac{11}{16(s + 4)} \right]
\]

\[
y = \frac{5}{16} + \frac{3}{4}t + \frac{11}{16}e^{-4t}
\]
2. (a) Let \( a \geq 0 \). We want to find the Laplace transform of the ramp function as given in the exercise. Since \( r_a(t) = 0 \) if \( t < a \), we compute:

\[
L^{-1}[r_a(t)] = \int_a^\infty k(t-a)e^{-st}dt
\]

\[
= k \int_a^\infty te^{-st}dt - ka \int_a^\infty e^{-st}dt.
\]

Compute the left integral using integration by parts (let \( u = t \) and \( dv = e^{-st}dt \)) and evaluate:

\[
k \int_a^\infty te^{-st}dt = k \left[ \frac{-te^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a = k \left( \frac{-ae^{-as}}{s} + \frac{e^{-as}}{s^2} \right).
\]

When we add this to the right integral, we get \( ke^{-as}(1 + as) + kae^{-as} = k e^{-as} \left( \frac{1}{s^2} + \frac{2a}{s} \right) \). Note: This integral is fiddly, so don’t take any mistakes in this solution personally.

(b) The graph of this function is constant \( t = 0 \) along the \( t \)-axis until the point \( t = a \) where it becomes an increasing linear function. (Big hint: The graph looks like a ramp.)

4. Find the inverse Laplace transform of the function \( \frac{e^{-2s}}{s-3} \).

\[
L^{-1} \left[ \frac{e^{-2s}}{s-3} \right] = u_2(t)e^{3(t-2)} = u_2(t)e^{3t-6}.
\]

6. Find the inverse Laplace transform of the function \( \frac{4e^{-2s}}{s(s+3)} \).

Write \( \frac{4}{s(s+3)} = \frac{4}{3s} - \frac{4}{3(s+3)} = \frac{4}{3} \left( \frac{1}{s} - \frac{1}{s+3} \right) \). So:

\[
L \left[ u_2(t)e^{0(t-2)} \right] = L[u_2(t)] = \frac{e^{-2s}}{s}, \text{ and } L \left[ u_2(t)e^{-3(t-2)} \right] = L[u_2(t)e^{6-3t}] = \frac{e^{-2s}}{s+3}
\]

So our function is \( \frac{4}{3} u_2(t) - \frac{4}{3} u_2(t)e^{6-3t} \).

8. \( \frac{dy}{dt} = u_2(t), \quad y(0) = 3 \)

Take the Laplace transform of both sides and plug in the initial condition to get \( sL[y] - 3 = \frac{e^{-2s}}{s} \). Then simplify to get \( L[y] = \frac{e^{-2s} + 3s}{s^2} \), which has partial fraction decomposition \( \frac{e^{-2s}}{s^2} + \frac{3}{s} \). Taking the inverse Laplace transform of both sides gives you \( y(t) = u_2(t)(t-2) + 3 \) as the solution of the IVP.

10. \( \frac{dy}{dt} + 7y = u_2(t), \quad y(0) = 3 \)

Take the Laplace transform of both sides and plug in the initial condition to get \( (s+7)L[y] - 3 = \frac{e^{-2s}}{s} \). Then simplify and decompose into partial fractions to get \( L[y] = \frac{e^{-2s}}{s(s+7)} = \frac{e^{-2s}}{s(s+7)} + \frac{3}{s + 7} \). Note that \( \frac{1}{s(s+7)} = \frac{1}{7s} - \frac{1}{7(s+7)} \).

So when we take the inverse Laplace transform of both sides we get \( y(t) = \frac{1}{7} u_2(t) - \frac{1}{7} u_2(t)e^{-7t+14} + 3e^{-7t} \).
12. \( \frac{dy}{dt} + y = 2u_3(t), \quad y(0) = 4 \)

Take the Laplace transform of both sides and plug in the initial condition to get \((s + 1)L[y] - 4 = \frac{2e^{-3s}}{s}\). Then simplify and decompose into partial fractions to get \(L[y] = \frac{2e^{-3s} + 4s}{s(s + 1)} = \frac{2e^{-3s}}{s(s + 1)} + \frac{4}{s + 1}\). Note that \(\frac{1}{s(s + 1)} = \frac{1}{s} - \frac{1}{s + 1}\). So when we take the Laplace transform of both sides we get \(y(t) = 2u_3(t) - 2u_3(t)e^{3-t} + 4e^{-t}\).
12. Complete the square for \( s^2 - 4s + 5 \).
   
   In this case, \( b = -4 \), and \( \left( s + \frac{b}{2} \right)^2 = (s - 2)^2 = s^2 - 4s + 4 \). So, \( s^2 - 4s + 5 = (s - 2)^2 + 1^2 \).

14. Complete the square for \( s^2 + 6s + 10 \).
   
   In this case, \( b = 6 \), and \( \left( s + \frac{b}{2} \right)^2 = (s + 3)^2 = s^2 + 6s + 9 \). So, \( s^2 + 6s + 10 = (s + 3)^2 + 1^2 \).

16. Compute the inverse Laplace transform of the given function using the result of Exercise 6.3.12: \( \frac{s}{s^2 - 4s + 5} \).
   
   \[
   \mathcal{L}^{-1}\left[ \frac{s}{s^2 - 4s + 5} \right] = \mathcal{L}^{-1}\left[ \frac{s}{(s - 2)^2 + 1^2} \right] = \mathcal{L}^{-1}\left[ \frac{s - 2 + 2}{(s - 2)^2 + 1^2} \right] = \mathcal{L}^{-1}\left[ \frac{s - 2}{(s - 2)^2 + 1^2} \right] + \mathcal{L}^{-1}\left[ \frac{2}{(s - 2)^2 + 1^2} \right] = e^{2t} \cos t + 2e^{2t} \sin t = e^{2t} (\cos t + 2 \sin t).
   \]

18. Compute the inverse Laplace transform of the given function using the result of Exercise 6.3.14: \( \frac{s + 1}{s^2 + 6s + 10} \).
   
   \[
   \mathcal{L}^{-1}\left[ \frac{s + 1}{s^2 + 6s + 10} \right] = \mathcal{L}^{-1}\left[ \frac{s + 1}{(s + 3)^2 + 1^2} \right] = \mathcal{L}^{-1}\left[ \frac{s + 3 - 2}{(s + 3)^2 + 1^2} \right] = \mathcal{L}^{-1}\left[ \frac{s + 3}{(s + 3)^2 + 1^2} \right] - \mathcal{L}^{-1}\left[ \frac{2}{(s + 3)^2 + 1^2} \right] = e^{-3t} \cos t - 2e^{-3t} \sin t = e^{-3t} (\cos t - 2 \sin t).
   \]

24. We want to use partial fractions and complex exponentials to compute the inverse Laplace transform of \( \frac{s}{s^2 - 4s + 5} \).
   
   First note that \( \frac{s}{s^2 - 4s + 5} = \frac{s}{(s - 2 - i)(s - 2 + i)} = \frac{1}{2 - i} + \frac{1}{s - 2 + i} \). So:
   
   \[
   \mathcal{L}^{-1}\left[ \frac{s}{s^2 - 4s + 5} \right] = \mathcal{L}^{-1}\left[ \frac{\frac{1}{2 - i}}{s - 2 - i} \right] + \mathcal{L}^{-1}\left[ \frac{\frac{1}{2 + i}}{s - 2 + i} \right] = \left( \frac{1}{2 - i} \right) e^{(2 + i)t} + \left( \frac{1}{2 + i} \right) e^{(2 - i)t} = e^{2t}(2 \sin(t) + \cos(t)).
   \]

26. We want to use partial fractions and complex exponentials to compute the inverse Laplace transform of \( \frac{s + 1}{s^2 + 6s + 10} \).
   
   First note that \( \frac{s + 1}{s^2 + 6s + 10} = \frac{s + 1}{(s + 3 - i)(s + 3 + i)} = \frac{\frac{1}{2 - i}}{s + 3 + i} + \frac{\frac{1}{2 + i}}{s + 3 - i} \). So:
   
   \[
   \mathcal{L}^{-1}\left[ \frac{s + 1}{s^2 + 6s + 10} \right] = \mathcal{L}^{-1}\left[ \frac{\frac{1}{2 - i}}{s + 3 + i} \right] + \mathcal{L}^{-1}\left[ \frac{\frac{1}{2 + i}}{s + 3 - i} \right] = \left( \frac{1}{2 - i} \right) e^{(-3 + i)t} + \left( \frac{1}{2 + i} \right) e^{(-3 - i)t} = e^{-3t} (\cos(t) - 2 \sin(t)).
   \]
28. \( \frac{d^2 y}{dt^2} - y = e^{2t} \), \( y(0) = 1 \), \( y'(0) = -1 \)

(a) \( s^2 L[y] - sy(0) + y'(0) - L[y] = L[e^{2t}] \)

(b) Substitute in the initial conditions and simplify to obtain the Laplace transform of the solution.

\[
\Rightarrow (s^2 - 1)L[y] = \frac{1}{s - 2} + s - 1
\]

\[
\Rightarrow L[y] = \frac{1}{(s - 2)(s^2 - 1)} + \frac{1}{s + 1}
\]

Now note that \( \frac{s}{s^2 - 1} = \frac{1}{2s + 1} + \frac{1}{2s - 1} \) and \( \frac{1}{s^2 - 1} = -\frac{1}{2} \frac{1}{s + 1} + \frac{1}{2} \frac{1}{s - 1} \). So:

\[
\Rightarrow L[y] = \frac{1}{3} \frac{1}{s - 2} - \frac{1}{3} \left( \frac{1}{2} \frac{1}{s + 1} + \frac{1}{2} \frac{1}{s - 1} \right) - \frac{2}{3} \left( -\frac{1}{2} \frac{1}{s + 1} + \frac{1}{2} \frac{1}{s - 1} \right) + \frac{1}{s + 1}
\]

\[
\Rightarrow L[y] = \frac{1}{3} \frac{1}{s - 2} + \frac{7}{6} \frac{1}{s + 1} - \frac{1}{2} \frac{1}{s - 1}
\]

(c) Take the inverse Laplace transform of both sides to get: \( y = \frac{1}{3} e^{2t} + \frac{7}{6} e^{-t} - \frac{1}{2} e^{t} \).

30. \( \frac{d^2 y}{dt^2} + 6 \frac{dy}{dt} + 13y = 13u_4(t), \) \( y(0) = 3 \), \( y'(0) = 1 \)

(a) \( s^2 L[y] - sy(0) + y'(0) + 6sL[y] - 6y(0) + 13L[y] = \frac{13e^{-4s}}{s} \)

(b) Substitute in the initial conditions and simplify to obtain the Laplace transform of the solution:

\[
s^2 L[y] - 3s + 1 + 6sL[y] - 18 + 13L[y] = \frac{13e^{-4s}}{s}
\]

\[
(s^2 + 6s + 13) L[y] = \frac{13e^{-4s}}{s} + 3s + 17
\]

\[
L[y] = \frac{\frac{13e^{-4s}}{s}}{s(s^2 + 6s + 13)} + \frac{3s + 17}{s^2 + 6s + 13}
\]

(c) Your final answer should simplify to:

\[
u_4(t) + \frac{1}{4} e^{(-3-2i)t} \left( (-2 + 3i)e^{12+4it}u_4(t) - (2 + 3i)e^{12+16i}u_4(t) + (6 - 8i)e^{4t+8i} + (6 + 8i)e^{8i} \right)
\]

32. \( \frac{d^2 y}{dt^2} + 3y = u_4(t) \cos(5(t - 4)) \), \( y(0) = 0 \), \( y'(0) = -2 \)

(a) \( s^2 L[y] - sy(0) + y'(0) + 3L[y] = e^{-4s} \left( \frac{s \cos(20) + 5 \sin(20)}{s^2 + 25} \right) \)

(b) Substitute in the initial conditions and simplify to obtain the Laplace transform of the solution:

\[
s^2 L[y] - 2 + 3L[y] = e^{-4s} \left( \frac{s \cos(20) + 5 \sin(20)}{s^2 + 25} \right)
\]

\[
(s^2 + 3) L[y] = e^{-4s} \left( \frac{s \cos(20) + 5 \sin(20)}{s^2 + 25} \right) + 2
\]

\[
L[y] = e^{-4s} \left( \frac{s \cos(20) + 5 \sin(20)}{(s^2 + 25)(s^2 + 3)} \right) + \frac{2}{s^2 + 3}
\]

(c) Your final answer should simplify to:

\[
\frac{1}{66} \left( u_4(t)(-3 \cos(4(t - 4)) \cos(20) + 3 \cos(\sqrt{3}(t - 4)) + (2 \sin(20 - 5t) + 4 \sqrt{3} \sin(\sqrt{3}(t - 4)) \sin(20) + 44 \sqrt{3} \sin(\sqrt{3}t) \right)
\]
2. \( \frac{d^2y}{dt^2} + 3y = 5\delta_2(t), \quad y(0) = 0, \quad y'(0) = 0 \)

Apply the Laplace transform to both sides and use the fact that \( L[\delta_2] = e^{-2s} \) to get:

\[
\begin{align*}
    s^2L[y] - sy(0) - y'(0) + 3L[y] &= 5e^{-2s} \\
    (s^2 + 3)L[y] &= 5e^{-2s} \\
    L[y] &= \frac{5e^{-2s}}{s^2 + 3} \\
    y &= \frac{5u_2(t)\sin(\sqrt{3}(t - 2))}{\sqrt{3}}.
\end{align*}
\]

4. \( \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = -2\delta_2(t), \quad y(0) = 2, \quad y'(0) = 0 \)

Apply the Laplace transform to both sides and use the fact that \( L[\delta_2] = e^{-2s} \) to get:

\[
\begin{align*}
    s^2L[y] - sy(0) - y'(0) + 2sL[y] - 2y(0) + 2L[y] &= -2e^{-2s} \\
    s^2L[y] - 2s + 2sL[y] - 4 + 2L[y] &= -2e^{-2s} \\
    (s^2 + 2s + 2)L[y] &= -2e^{-2s} + 2s + 4 \\
    L[y] &= \frac{-2e^{-2s} + 2s + 4}{s^2 + 2s + 2} \\
    \frac{1}{2}L[y] &= \frac{-e^{-2s} + s + 2}{(s + 1)^2 + 1} \\
    \frac{1}{2}L[y] &= \frac{-e^{-2s} + s + 1 + 1}{(s + 1)^2 + 1} \\
    \frac{1}{2}L[y] &= \frac{-e^{-2s}}{(s + 1)^2 + 1} + \frac{s + 1}{(s + 1)^2 + 1} + \frac{1}{(s + 1)^2 + 1} \\
    y &= 2e^{-t}u_2(t)\sin(2 - t) + 2e^{-t}\sin(t) + 2e^{-t}\cos(t) \\
    y &= 2e^{-t}(u_2(t)\sin(2 - t) + \sin(t) + \cos(t))
\end{align*}
\]