Instructions. Complete each of the problems below.

Q1 [15 points] Consider the system \( \frac{dY}{dt} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} Y. \)

(a) Find the eigenvalues.
\[
\det \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i
\]

(b) Determine if the origin is a spiral sink, a spiral source, or a center.

(c) Determine the natural period and natural frequency of the oscillations (clearly label each).

\[ \text{Natural Period: } T = \frac{\pi}{2} \quad \text{Natural frequency: } \omega = \frac{\pi}{10}. \]

(d) Determine the direction of oscillations in the phase plane.

\( \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \). No clockwise.

(e) Find the general solution.

\[
Y(t) = e^{2it} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{2it} \left( \cos(2t) + i \sin(2t) \right) \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos(2t) + i \sin(2t) \\ -2i \cos(2t) \end{pmatrix}
\]

\[
Y(t) = \begin{pmatrix} \cos(2t) \\ -2i \cos(2t) \end{pmatrix} + i \begin{pmatrix} \sin(2t) \\ -2 \sin(2t) \end{pmatrix} \Rightarrow Y(t) = c_1 \begin{pmatrix} \cos(2t) \\ -2i \cos(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) \\ -2 \sin(2t) \end{pmatrix}
\]

(f) Solve the solution with the initial condition \( Y_0 = (1, 0) \).

\[
k_1 = 1, \quad k_2 = 0 \Rightarrow Y(t) = \begin{pmatrix} \cos(2t) \\ -2i \cos(2t) \end{pmatrix}
\]
Q2 [15 points] Consider the system \( \frac{dY}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} Y. \)

Find the single eigenvalue and its eigenvector.

\[
\text{det} \begin{pmatrix} -2 - \lambda & -1 \\ 1 & -4 - \lambda \end{pmatrix} = 0 \implies - \lambda (2+4\lambda) -1 + 8 + 2\lambda + 4\lambda + \lambda^2 + 1 = 0 \\
\implies \lambda^2 + 6\lambda + 9 = 0 \\
\implies (\lambda + 3)^2 = 0 \\
\implies \lambda = -3
\]

\[
\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{cases} x - y = 0 \Rightarrow y = x \\
x^2 + 4y^2 = 9 \\ \end{cases}
\]

Sketch the phase portrait.

Find the general solution.

\( V_0 = (x_0, y_0). \)

\( V_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_0, y_0 \end{pmatrix} \)

\( v(t) = e^{-3t} v_0 + te^{-3t} v_1 = e^{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} + te^{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} \)

State the solution with initial condition \((x_0, y_0) = (1, 0)\).

\( v(t) = (1) e^{-3t} + te^{-3t} (1) \).
Q3 [15 points] Consider the system \( \frac{dY}{dt} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} Y \).

Find the eigenvalues and their eigenvectors.

\[
\det \begin{pmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0 \Rightarrow (4-\lambda)(1-\lambda) - 4 = 0 \\
\Rightarrow 4 - 4\lambda + \lambda^2 - 4 = 0 \\
\Rightarrow \lambda^2 - 5\lambda = 0 \\
\Rightarrow \lambda(\lambda - 5) = 0 \\
\Rightarrow \lambda = 0, 5.
\]

\( \lambda = 0 \)

\[
\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4x + 2y = 0 \\ 2x + y = 0 \end{cases} \\
y = -2x, [\begin{pmatrix} 1 \\ -2 \end{pmatrix}]
\]

\( \lambda = 5 \)

\[
\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4x + 2y = 0 \\ 2x + y = 0 \end{cases} \\
x - 2y = 0, \left[ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]
\]

Sketch the phase portrait.

Find the general solution.

\[
Y(t) = k_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} 
\]

\[
\begin{align*}
k_1 + 2k_2 &= 1 \\
-2k_1 + k_2 &= 0
\end{align*}
\]

\[
\frac{2}{5} = 2k_1 \Rightarrow k_1 = \frac{1}{5},
\]

\[
k_2 = \frac{5}{2}.
\]

State the solution with initial condition \((x_0, y_0) = (1, 0)\).

\[
Y(t) = \frac{1}{5} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{2}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}.
\]