Intro to Differential Equations  
MATH 2070  
Partial Fractions Problems

Discussion

We all know that

\[ \int \left( \frac{1}{p} - \frac{1}{p-1} \right) \, dp = \ln |p| - \ln |p-1| + C \]

\[ = \ln \left| \frac{p}{p-1} \right| + C \]

Suppose we wanted to compute the integral \( \int \frac{1}{p-p^2} \, dp \). One way to do it is to “realize” that

\[ \frac{1}{p-p^2} = \frac{1}{p} - \frac{1}{p-1} \quad (\star) \]

and to use the method described above. Here’s how we could come to this realization.

**Step 1**  Factor \( p-p^2 = p(1-p) \).

**Step 2**  Write \( \frac{1}{p-p^2} = \frac{A}{p} + \frac{B}{p-1} \). We want to find the constants \( A \) and \( B \).

**Step 3**  Solve for \( A \) and \( B \). Here’s one way to do it. Multiply both sides of the equation by \( p(1-p) \) to get

\[ 1 = A(1-p) + Bp \]

Since this must hold for all \( p \), we can take

- \( p = 1 \) and get \( 1 = A \cdot 0 + B \cdot 1 \) so \( B = 1 \).
- \( p = 0 \) and get \( 1 = A \cdot 1 + B \cdot 0 \) so \( A = 1 \).

We conclude that

\[ \frac{1}{p-p^2} = \frac{1}{p} + \frac{1}{1-p} \]

which is equivalent to what we displayed before in equation \((\star)\).

This is not meant to be a complete discussion of the method of partial fractions or the kinds of algebra that must be done to find the unknown constants. Please consult your calculus text for more complete information.

On the next page, I’ve included a few problems and “suggest” the splitting that must take place.
Problems

1. $\int \frac{1}{p + p^2} \, dp$. Write \[ \frac{1}{p + p^2} = \frac{A}{p} + \frac{B}{1 + p}. \]

2. $\int \frac{1}{p^2 + 3p} \, dp$. Write \[ \frac{1}{p^2 + 3p} = \frac{A}{p} + \frac{B}{3 + p}. \]

3. $\int \frac{1}{p^3 + 4p^2} \, dp$. Write \[ \frac{1}{p^3 + 4p^2} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p + 4}. \]

Notice that even though the denominator factors as $p^2 (p + 4)$, we still need to allow the possibility that the factors $p$, $p^2$ and $p + 4$ can appear in the denominator.

4. $\int \frac{x + 1}{x(x^2 + 1)} \, dx$. Write \[ \frac{x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}. \]

Notes: (1) Observe that $x^2 + 1$ doesn't factor further (at least over the real numbers).
(2) Recall that
\[ \int \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \ln (x^2 + 1) + C \]
and that
\[ \int \frac{1}{x^2 + 1} \, dx = \arctan (x) + C \]