Math 361, Problem set 3

Due 9/20/10

1. (1.4.21) Suppose a fair 6-sided die is rolled 6 independent times. A match occurs if side \( i \) is observed during the \( i \)th trial, \( i = 1, \ldots, 6 \).

   (a) What is the probability of at least one match during on the 6 rolls.

   (b) Extend part (a) to a fair \( n \)-sided die with \( n \) independent rolls. Then determine the limit of the probability as \( n \to \infty \).

   Answer: It is much easier to compute the probability that ever roll is a non-match. For any given roll, the probability of a non-match is \( \frac{5}{6} \). Since the rolls are independent, the probability that there are no matches is \( (\frac{5}{6})^6 \). Therefore the probability that there is at least one match is \( 1 - (\frac{5}{6})^6 \).

   Likewise for any roll in the general case the probability of a non-match is \( \frac{n-1}{n} \), and hence the probability of at least one match is

   \[
   1 - \left(\frac{n-1}{n}\right)^n.
   \]

   Since \( \lim_{n \to \infty} (1 - 1/n)^n = e^{-1} \), in the limit this is \( 1 - e^{-1} \).

2. (1.4.32) Hunters \( A \) and \( B \) shoot at a target; their probabilities of hitting the target are \( p_1 \) and \( p_2 \) respectively. Assuming independent, can \( p_1 \) and \( p_2 \) be chosen so that

   \[ P(0 \text{ hits}) = P(1 \text{ hit}) = P(2 \text{ hits}) \]

   Answer: It is not possible. We will show this by contradiction, suppose it was possible. Since there is either zero or one or two hits, then \( P(0 \text{ hits}) = P(1 \text{ hit}) = P(2 \text{ hits}) = \frac{1}{3} \). Then

   \[ 0 = P(2 \text{ hits}) - P(0 \text{ hits}) = p_1 p_2 - (1 - p_1)(1 - p_2) = p_1 + p_2 - 1. \]

   Therefore \( p_2 = 1 - p_1 \). In order for \( p_1 p_2 = \frac{1}{3} \),

   \[
   p_1 (1 - p_1) = \frac{1}{3}.
   \]
However, the function $f(x) = x(1 - x)$ is maximized at $x = 1/2$. Therefore there are no solutions to this equation, and no such $p_1, p_2$ exist. (Alternately, one could use the quadratic formula here to show that the roots are complex.)

3. (1.5.1) Let a card be selected from an ordinary deck of playing cards. The outcome $c$ is one of these 52 cards. Let $X(c) = 4$ if $c$ is an ace, let $X(c) = 3$ if $c$ is a king, $X(c) = 2$ if $c$ is a king and $X(c) = 1$ if $c$ is a jack. Otherwise $X(c) = 0$. Suppose $P$ assigns a probability of $\frac{1}{52}$ to each outcome $c$. Describe the induced probability $P_X(D)$ on the space $D = \{0, 1, 2, 3, 4\}$ of the random variable $X$.

Answer

$$P_X(\{k\}) = \begin{cases} \frac{4}{52}, k \in \{1, 2, 3, 4\} \\ \frac{36}{52}, k = 0 \end{cases}$$

4. (1.5.9) Consider an urn which contains slips of paper each with one of the numbers $1, 2, \ldots, 100$ on it. Suppose there are $i$ slips with the number $i$ on it for $i = 1, 2, \ldots, 100$. E.g. there are 25 slips of paper with the number 25. Suppose one slip is drawn at random, let $X$ be the number of the slip.

(a) Show that $X$ has pmf $p(x) = x/5050, x = 1, 2, 3, \ldots, 100$, zero elsewhere.

(b) Compute $P(X \leq 50)$.

(c) Show that the cdf of $X$ is $F(x) = \lfloor x \rfloor (\lfloor x \rfloor + 1) / 10100$ for $1 \leq x \leq 100$ where $\lfloor x \rfloor$ is the greatest integer in $x$ (ie, $\lfloor 100.12 \rfloor = 100$.)

Answer: To answer (a) it suffices to note that the number of slips is

$$\sum_{n=1}^{100} n = \frac{(100)(101)}{2} = 5050.$$  

Then it is clear that $p(x) = \frac{x}{5050}$ as there are $x$ slips out of 5050 with the number $x$.

For $b$ we need to compute

$$P(X \leq 50) = \sum_{n=1}^{50} p(n) = \sum_{n=1}^{50} \frac{n}{5050} = \frac{(50)(51)}{10100}.$$  

For (c) we need to compute

$$F(x) = P(X \leq x) = P(X \leq \lfloor x \rfloor) = \sum_{n=1}^{\lfloor x \rfloor} \frac{n}{5050} = \frac{\lfloor x \rfloor (\lfloor x \rfloor + 1)}{10100}.$$  

Note that the greatest integer just enters the picture here because $F(x)$ is defined for all real numbers, but $F(x)$ is a step function, only changing at integer boundaries.
5. (1.5.10) Let $X$ be a random variable with space $\mathcal{D}$. For a sequence of sets $\{D_n\}$ in $\mathcal{D}$ show that

$$\{c : X(c) \in \bigcup_n D_n\} = \bigcup_n \{c : X(c) \in D_n\}$$

Use this to show that the induced probability $P_X$ (see eq. 1.5.1) satisfies the third (additive) axiom of probability.

**Answer:** Let $A = \{c : X(c) \in \bigcup_n D_n\}$, $B_n = \{c : X(c) \in D_n\}$ and $B = \bigcup_n \{c : X(c) \in D_n\} = \bigcup_n B_n$. We show $A \subseteq B$ and $B \subseteq A$.

Suppose $c \in A$. Then $X(c) \in \bigcup_n D_n$, and hence $X(c) \in D_j$ for some $j$. By definition, this means that $c \in B_j$. However, then $c \in B = \bigcup_n B_n$. Therefore $A \subseteq B$.

Next we show $B \subseteq A$. Suppose $c \in B = \bigcup_n B_n$. Then $c \in B_j$ for some $j$. Therefore $X(c) \in D_j \subseteq \bigcup_n D_n$. Thus $c \in A$, by definition. Therefore $A = B$.

Suppose that $D_1, D_2, \ldots$ are disjoint subsets of $\mathcal{D}$. Then it is clear that the sequence $B_1, B_2, \ldots$ are also disjoint. Therefore

$$P_X(\bigcup_n D_n) = P(A) = P(\bigcup_n B_n) = \sum_n P(B_n) = \sum_n P_X(D_n).$$

where the second equality on the first line is what we showed above, the second line is because $P$ is a probability set function, and the first and last equalities are the definition of $P_X$.

6. (1.6.2) Let a bowl contain 10 chips of the same shape and size. One, and only one, of these chips is red. Continue to draw chips from the bowl, one at a time and at random without replacement, until the red chip is drawn.

(a) Find the pmf of $X$, the number of trials needed to draw the red chip.

(b) Compute $P(X \leq 4)$.

**Answer:**

This is the smart drunk problem in surprise, and hence (see last solutions) $p(x) = \frac{1}{10}$ for $x = 1, \ldots, 10$ with $p(x) = 0$ otherwise. Thus $P(X \leq 4) = \frac{4}{10}$.  

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