1. (2.1.5) Given that the nonnegative function \( g(x) \) has the property that 
\[ \int_0^\infty g(x)dx = 1, \]
show that
\[
f(x_1, x_2) = \frac{2g(\sqrt{x_1^2 + x_2^2})}{\pi \sqrt{x_1^2 + x_2^2}}, \quad 0 < x_1 < \infty \quad 0 < x_2 < \infty,
\]
zero elsewhere, satisfies the conditions for a pdf of two continuous-type random variables \( X_1 \) and \( X_2 \). **Hint: Use polar coordinates**

**Answer:** \( f(x_1, x_2) \geq 0 \) as the ratio of two non-negative functions.

We do the change of variables \( x_1 = r \cos(\theta) \) and \( x_2 = r \sin(\theta) \); the Jacobian of this change of variables is \( r \). Thus
\[
\int \int f(x_1, x_2)dx_1dx_2 = \int_0^\infty \int_0^{\pi/2} 2g(r) \frac{\pi r}{2} drd\theta
\]
\[
= \int_0^\infty g(r)dr = 1.
\]
so \( f(x_1, x_2) \) satisfies the conditions for a joint PDF of \( X_1 \) and \( X_2 \).

2. (2.1.8) Let 13 cards be taken, at random and without replacement, from an ordinary deck of playing cards. If \( X \) is the number of spades in these 13 cards, find the pmf of \( X \). If, in addition \( Y \) is the number of heardts in these 13 cards, find the probability \( P(X = 2, Y = 5) \). What is the joint pmf of \( X \) and \( Y \). **Answer:**

We have
\[
p_X(x) = \binom{13}{x} \binom{39}{13-x} \binom{52}{13}/\binom{52}{13}
\]
\[
p_{X,Y}(x, y) = \binom{13}{x} \binom{10}{y} \binom{26}{13-x-y} \binom{52}{13}/\binom{52}{13}.
\]
and
\[ P(X = 2, Y = 5) = P(X, Y)(2, 5) = \binom{13}{5} \binom{26}{6} \binom{6^2}{13} \]

3. (2.1.14) Let \( X_1, X_2 \) be two random variables with joint pmf \( p(x_1, x_2) = (1/2)^{x_1+x_2} \) for \( x_i \in \{1, 2, 3, 4, \ldots\} \) with \( i = 1, 2 \) and zero elsewhere. Determine the joint mgf of \( X_1, X_2 \). Show that \( M(t_1, t_2) = M(t_1, 0)M(0, t_2) \).

Answer

\[
M(t_1, t_2) = \mathbb{E}[e^{t_1 X_1 + t_2 X_2}] = \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \frac{1}{2} e^{t_1 x_1 + t_2 x_2}
\]
\[
= \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \left( \frac{1}{2} e^{t_1/2} \right)^{x_1} \left( \frac{1}{2} e^{t_2/2} \right)^{x_2}
\]
\[
= \sum_{x_1=1}^{\infty} \left( \frac{1}{2} e^{t_1/2} \right)^{x_1} \left( \frac{e^{t_2/2}}{1 - e^{t_2/2}} \right)
\]
\[
= \left( \frac{e^{t_2/2}}{2 - e^{t_2}} \right) \left( \frac{e^{t_1}}{2 - e^{t_1}} \right)
\]

so long as \( t_1 < \ln(2) \) and \( t_2 < \ln(2) \) so that the geometric series converge. That \( M(t_1, t_2) = M(t_1, 0)M(0, t_2) \) is clear.

4. (2.1.16) Let \( X \) and \( Y \) have the joint pdf \( f(x, y) = 6(1 - x - y) \) for \( x + y < 1 \), \( 0 < x, 0 < y \) and zero elsewhere. Compute \( P(2X + 3Y < 1) \) and \( \mathbb{E}[XY + 2X^2] \).

Answer:

\[
P(2X + 3Y < 1) = \int_0^{1/2} \left( \int_0^{(1-2x)/3} 6(1 - x - y) dy \right) dx
\]
\[
= 6 \int_0^{1/2} (y - xy - y^2/2)_{y=0}^{(1-2x)/3} dx
\]
\[
= \int_0^{1/2} 5x/3 - 14x/3 + 8x^2/3 dx
\]
\[
= \frac{5x}{3} - \frac{7x^2}{3} + \frac{8x^2}{9} \bigg|_0^1 = \frac{13}{36}
\]

\[
\mathbb{E}[XY + 2X^2] = \int_0^1 \left( \int_0^{1-x} (xy + 2x^2) 6(1 - x - y) dy \right) dx = \cdots = \frac{1}{4}
\]

Sorry, too lazy to type out the steps.
5. (2.2.2) Let $X_1$ and $X_2$ have the joint pmf $p(x_1, x_2) = \frac{x_1 x_2}{36}$ for $x_1 = 1, 2, 3$ and $x_2 = 1, 2, 3$; zero elsewhere. Find first the joint pmf of $Y_1 = X_1 X_2$ and $Y_2 = X_2$, and then find the marginal pmf of $Y_1$.

$\text{Answer}$

$P_{Y_1, Y_2}(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2) = P(X_1 X_2 = y_1, X_2 = y_2) = \frac{y_1}{36}$.

for $y_2 = 1, 2, 3$ and $y_1 = y_2, 2y_2, 3y_3$; zero otherwise.

$P_{Y_1}(y_1) = \sum_{y_2} P_{Y_1, Y_2}(y_1) = \begin{cases} \frac{y_1}{36} & y_1 = 1, 4, 9, \\ \frac{y_1}{36} & y_1 = 2, 3, 6. \end{cases}$

6. (2.2.7) Use the formula (2.2.1) to find the pdf of $Y_1 = X_1 + X_2$, where $X_1$ and $X_2$ have the joint pdf $f_{X_1, X_2}(x_1, x_2) = 2e^{-(x_1 + x_2)}$, $0 < x_1 < x_2 < \infty$, zero elsewhere.

$\text{Answer:}$

$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1, X_2}(y_1 - y_2, y_2) dy_2$

$= \int_{y_1/2}^{y_1} 2e^{-y_1} dy = y_1 e^{-y_1}$

for $y_1 > 0$. Here the bounds arise as $y_1 - y_2 < y_2$, so $y_2 > y_1/2$ and $y_1 - y_2 > 0$, so $y_2 < y_1$. 

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