1. (6.1.11) Let $X_1, \ldots, X_n$ be a random sample from an $N(\theta, \sigma^2)$ distribution, where $\sigma^2$ is fixed and known, and $-\infty < \theta < \infty$.

(a) Show that the mle of $\theta$ is $\bar{X}$.

(b) If $\theta$ is restricted by $0 \leq \theta < \infty$, show that the mle of $\theta$ is $\hat{\theta} = \max\{0, \bar{X}\}$.

Answer: We have that
\[ L(\theta) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n \exp\left(-\frac{1}{2\sigma} \sum (X_i - \theta)^2\right) \]
so
\[ \ell(\theta) = -\frac{n}{2} \log(2\pi\sigma) - \frac{1}{2\sigma^2} \sum (X_i - \theta)^2 \]
and
\[ \ell'(\theta) = -\frac{1}{\sigma^2} \sum (X_i - \theta) \]
Setting $\ell'(\theta) = 0$, we see that $\hat{\theta} = \bar{X}$.

For (b), we observe that $\ell'(\theta) < 0$ for $\theta < \bar{X}$ and $\ell'(\theta) < 0$ for $\theta > \bar{X}$. Thus if we know that $\theta \geq 0$, and $\bar{X} < 0$, $\ell(\theta)$ is maximized at 0. That is $\hat{\theta} = \max\{0, \bar{X}\}$.

2. Let $X_1, \ldots, X_n$ be a random sample from an $N(0, \theta)$ distribution. We want to estimate the standard deviation $\sqrt{\theta}$. Find the constant $c$ so that $Y = c \sum |X_i|$ is an unbiased estimator of $\sqrt{\theta}$ and determine it’s efficiency.

We note that
\[ \mathbb{E}[|X_i|] = 2 \int_0^\infty \frac{1}{\sqrt{2\pi\theta}} x e^{-x^2/2\theta} \, dx \]
\[ = 2 \int_0^\infty \frac{\sqrt{\theta}}{\sqrt{2\pi}} e^{-u} \, du = \sqrt{2\theta/\pi}. \]

Thus we take $c = \frac{\sqrt{\pi}}{\sqrt{2\pi}}$. 

We compute $I(\theta)$ next. We have 

$$f(x; \theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2\theta} x^2\right)$$

$$\log(f(x; \theta)) = -\frac{1}{2} \log(2\pi\theta) - \frac{1}{2\theta} x^2$$

$$\frac{\partial \log(f(x; \theta))}{\partial \theta} = -\frac{1}{2\theta} + \frac{x^2}{2\theta^2}$$

$$\frac{\partial^2 \log(f(x; \theta))}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}.$$ 

Thus 

$$I(\theta) = \mathbb{E}[-\frac{\partial^2 \log(f(X; \theta))}{\partial \theta^2}] = \mathbb{E}[\frac{X^2}{\theta} - \frac{1}{2\theta^2}] = \frac{1}{2\theta^2},$$

as $\mathbb{E}[X^2] = \theta$.

We have $k(\theta) = \sqrt{\theta}$, and $k'(\theta) = \frac{1}{2\sqrt{\theta}}$. Thus the Rao-Cramer lower bound is 

$$\frac{(k'(\theta))^2}{nI(\theta)} = \frac{\theta}{2n}.$$ 

On the other hand, 

$$\text{Var}(Y) = c^2 n \text{Var}(|X_i|) = c^2 n (\mathbb{E}[X_i^2] - \mathbb{E}[|X_i|^2]) = \theta n \left(\frac{\pi - 2}{2}\right).$$ 

Thus we see that the efficiency is 

$$\frac{(k'(\theta))^2}{\text{Var}(Y)nI(\theta)} = \frac{1}{\pi - 2}.$$ 

3. (6.2.14) Let $S^2$ be the sample variance of a random sample of size $n > 1$ from $N(\mu, \theta)$, $0 < \theta < \infty$, where $\mu$ is known. We know $\mathbb{E}[S^2] = \theta$.

(a) What is the efficiency of $S^2$?

(b) Under these conditions, what is the mle $\hat{\theta}$ of $\theta$?

(c) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$.

**Answer:** 

$$f(x; \theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2\theta} (x - \mu)^2\right)$$

$$\log(f(x; \theta)) = -\frac{1}{2} \log(2\pi\theta) - \frac{1}{2\theta} (x - \mu)^2$$

$$\frac{\partial \log(f(x; \theta))}{\partial \theta} = -\frac{1}{2\theta} + \frac{(x - \mu)^2}{2\theta^2}$$

$$\frac{\partial^2 \log(f(x; \theta))}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{(x - \mu)^2}{\theta^3}.$$
and as before \( I(\theta) = \frac{1}{2\theta^2} \), and \( k(\theta) = \theta \), so \( k'(\theta) = 1 \). On the other hand, since \( S^2 \) has a \( \chi^2(n-1) \) distribution, we know that \( \text{Var}(S^2) = \frac{\sigma^2}{(n-1)} \text{Var}((n-1)S^2/\theta) = \frac{2\theta^2}{n-1} \). We have that the efficiency of \( S^2 \) is
\[
\frac{2\theta^2}{n\text{Var}(S^2)} = \frac{n-1}{n}.
\]

We have
\[
\ell(\theta) = -\frac{n}{2} \log(2\pi\theta) - \frac{1}{2\theta} \sum (X_i - \mu)^2 \\
\ell'(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum (X_i - \mu)^2.
\]

Solving, we see \( \hat{\theta} = \frac{1}{n} \sum (X_i - \mu)^2 \).

For (c), we see that \( \sqrt{n}(\hat{\theta} - \theta) \rightarrow N(0, 1/I(\theta)) = N(0, 2\theta^2) \).

4. (6.3.5) Let \( X_1, \ldots, X_n \) be a random sample from a \( N(\mu_0, \theta) \) distribution, where \( 0 < \theta < \infty \) and \( \mu_0 \) is known. Show that the likelihood ratio test of \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta \neq \theta_0 \) can be based upon the statistic
\[
W = \sum_{i=1}^{n} (X_i - \mu_0)^2/\theta_0.
\]
Determine the null distribution of \( W \) (that is, the distribution of \( W \) given that \( \theta = \theta_0 \), and give, explicitly a rejection rule for a level \( \alpha \) test.

**Hint/Note:** If \( \theta = \theta_0 \), so that \( X_i \sim N(\mu_0, \theta) \) what is the distribution of \( (X_i - \mu_0)^2/\theta_0 \)? It’s one we know. Maybe figure out the distribution of \( (X_i - \mu_0)/\sqrt{\theta_0} \) first.

**Answer:**

We have
\[
L(\theta) = \left( \frac{1}{\sqrt{2\pi\theta}} \right)^{n/2} \exp \left( -\frac{1}{2\theta} \sum (x_i - \mu)^2 \right).
\]

Thus
\[
L(\theta_0) = \left( \frac{1}{\sqrt{2\pi\theta_0}} \right)^{n/2} e^{-W/2}.
\]

We found \( \hat{\theta} \) in the last problem, so
\[
L(\hat{\theta}) = \left( \frac{1}{\sqrt{2\pi \frac{1}{n} \sum (x_i - \mu)^2}} \right)^{n/2} e^{-n/2}
\]

Combining, we have
\[
\Lambda = n^{-n/2}e^{n/2}W^{n/2}e^{-W/2},
\]
so this test depends on \( W \) as desired. Note that \( \Lambda \leq c \) is the same as \( W \leq c_1 \) or \( W \geq c_2 \) for some constants \( c_1 \) and \( c_2 \). Since \( W \sim \chi^2(n) \), we take \( c_1 = \chi^2_{1/2}(n) \) and \( c_2 = \chi^2_{1-\alpha/2}(n) \) to get a test of size \( \alpha \).
5. (6.3.8) Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean $\theta > 0$.

(a) Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is based upon the statistic $Y = \sum_{i=1}^{n} X_i$. Obtain the null distribution of $Y$.

(b) For $\theta_0 = 2$ and $n = 5$, find the significance level of the test that rejects $H_0$ if $Y \leq 4$, or $Y \geq 17$.

*Note:* For (a), show that the test is of the form reject $H_0$ if $f(Y) > c$. It will not immediately look like it is of the form $Y > c$. The null distribution of $Y$ is the distribution of $Y$ if the null hypothesis is true.

*Answer:* We have

$$L(\theta) = e^{-n\theta} \theta^{\sum X_i} \prod X_i.$$ 

We also know $\hat{\theta} = \bar{X}$, so

$$\Lambda = e^{-n\theta_0} (\theta_0)^Y e^Y \frac{(Y/n)^Y}{(Y/n)^Y}.$$ 

This is a function of $Y$; which is $Poisson(n\theta)$.

For (b), we have that $Y \sim Poisson(10)$, and hence the size of this test is $(.029) + (1 - .973)$. 
