Show that if \((x_n)\) is a bounded sequence which does not converge, then \(x_n\) contains two convergent subsequences with different limits. (HINT: \((x_n)\) contains a convergent subsequence, say with limit \(L\), by Bolzano-Weierstrass. But \(x_n\) itself does not converge to \(L\), so it contains “many” terms “not close” to \(L\). I am being deliberately vague about definitions of “many” and “not close” so that you can decide on the proper interpretation.)

Suppose that \((x_n)\) and \((y_n)\) are bounded sequences. Show that there exists an increasing sequence of integers \(n_k\) so that the subsequences \((x_{n_k})\) and \((y_{n_k})\) both converge. (NOTE: It is clear from Bolzano-Weierstrass that each of the sequences \((x_n)\) and \((y_n)\) has a convergent subsequence, but they might involve different indices; for instance maybe \(x_1, x_3, x_5, \ldots\) and \(y_2, y_4, y_6, \ldots\) both converge. The problem asks for ONE sequence of indices \((n_k)\) so that both \((x_{n_k})\) and \((y_{n_k})\) converge.)