2.1.3. If \( X \) and \( Y \) are SFTs on an alphabet \( A \), then there exist finite sets \( F_1, F_2 \subset A^* \) so that \( X = X_{F_1} \) and \( Y = X_{F_2} \). Recall that we showed in Exercise 1.2.5 that \( X_{F_1} \cap X_{F_2} = X_{F_1 \cup F_2} \). Therefore, \( X \cap Y = X_{F_1 \cup F_2} \), which is an SFT since the union of two finite sets is finite.

\[ X \cup Y \] need not be an SFT; for instance, clearly \( X = \{0,1\}^\mathbb{Z} \) and \( Y = \{1,2\}^\mathbb{Z} \) are SFTs, but their union is not. Suppose for a contradiction that \( X \cup Y \) is an SFT; then it is \( m \)-step for some \( m \). \( 01^m \) is part of the biinfinite sequence \( 0^\infty 1^\infty \in X \subset X \cup Y \), so \( 01^m \in L(X \cup Y) \). Also, \( 1^m 2 \) is part of the biinfinite sequence \( 1^\infty 2^\infty \in Y \subset X \cup Y \), so \( 1^m 2 \in L(X \cup Y) \). But, clearly \( 01^m 2 \) is not a subword of any sequence in \( X \) or \( Y \), so \( 01^m 2 \notin L(X \cup Y) \). This contradicts the fact that in an \( m \)-step SFT, any words in the language which overlap in \( m \) symbols can be "glued" to make a new word in the language. Our original assumption was therefore wrong, and \( X \cup Y \) is not an SFT.

2.1.5. \( \implies \): If \( u \in L_n(X) \), then since \( L(X) \) is factorial, every subword of \( u \) is in \( L(X) \).

\( \impliedby \): Suppose that every subword of \( u \) which has length \( m + 1 \) is in \( L(X) \). Then, if \( u = u_1 \ldots u_n, u_1 \ldots u_{m+1} \) and \( u_2 \ldots u_{m+2} \) are in \( L(X) \) by assumption. Then since \( X \) is an \( m \)-step SFT, they can be "glued" to show that \( u_1 \ldots u_{m+2} \in L(X) \). Then \( u_3 \ldots u_{m+3} \) is in \( L(X) \) by assumption, and we can "glue \( u_1 \ldots u_{m+2} \) and \( u_3 \ldots u_{m+3} \) to show that \( u_1 \ldots u_{m+3} \in L(X) \). Continuing in this fashion will show that \( u \in L(X) \).

2.1.9. Claim: \( X(S) \) is an SFT if and only if \( S \) is finite or cofinite.

\( \impliedby \): If \( S \) is cofinite, then \( S^c \) is finite, and it is not hard to check that \( \{10^n 1 : n \in S^c \} \) is a finite forbidden list inducing \( X(S) \), so \( X(S) \) is an SFT. If \( S \) is finite, then denote by \( N \) the maximal element of \( S \). It is not hard to check that \( \{10^n 1 : n \in S^c \cap [1,N] \} \cup \{0^{N+1}\} \) is a finite forbidden set inducing \( X(S) \), and so \( X(S) \) is an SFT.

\( \implies \): Assume that \( S \) is not finite, and that \( X(S) \) is an SFT. Then \( X(S) \) is \( m \)-step for some \( m \). Since \( S \) is not finite, there exists \( k \in S \) greater than \( m \). Then since the biinfinite sequence \( \ldots 10^k 10^k 10^k \ldots \) is in \( X(S) \) and \( k > m \), \( 10^m, 0^{m+1}, \) and \( 0^{m+1} \) are all in \( L(X(S)) \). Then, by "gluing" \( 0^{m+1} \) with itself, \( 0^{m+2} \) is also in \( L(X(S)) \), and we can continue in this way to show that \( 0^n \) is in \( L(X(S)) \) for any \( n > m \). Then by "gluing" \( 10^n, 0^n, \) and \( 0^m 1 \), we see that \( 10^n 1 \in L(X(S)) \) for any \( n > m \). By definition of the \( S \)-gap shift, this implies that \( n \in S \) for all \( n > m \), implying that \( S \) is cofinite. We have shown that if \( S \) is not finite, it is cofinite, and therefore it is either finite or cofinite.

2.2.3. If \( H \) is a subgraph of \( G \), then \( A_H \) is obtained from \( A_G \) by deleting the \( i \)th
row and $i$th column for all $i$ in some (possibly empty) set $I$ and then decreasing some (possibly empty) subset of the entries. The deletion of rows and columns corresponds to removal of vertices, and the decreasing of entries corresponds to removal of edges.

2.2.10. $(a) \implies (b)$: If all paths in $G$ have length less than $n$, then for every $i, j \in V(G)$, there are no paths from $i$ to $j$ of length $n$, implying by Proposition 2.2.12(a) that $a_{ij}^{(n)} = 0$ for all $i, j$, implying that $A^n = 0$.

$(b) \implies (c)$: If $A^n = 0$, then $a_{ij}^{(n)} = 0$ for all $i, j$, and so there are no paths in $G$ of length $n$. This clearly implies that there are no biinfinite paths in $G$, and so $X_G = \emptyset$.

$(c) \implies (a)$: If $X_G = \emptyset$, then there are no biinfinite paths in $G$. This implies that $G$ contains no cycles (otherwise you could follow the cycle indefinitely), which implies by the Pigeonhole Principle that $G$ contains no paths of length greater than $|V(G)|$. 

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