2.2.12. Recall that by Proposition 2.2.12(a), for any graph G with adjacency matrix $A$ and any positive integer $m$, $a_{ij}^{(m)}$ is the number of paths from vertex $i$ to vertex $j$ of length $m$. By definition, $G$ is irreducible iff for all $i,j \in V(G)$ there exists a path from $i$ to $j$ ($\iff$ for all $i,j \in V(G)$). Clearly this happens iff for all $i,j \in V(G)$, there exists $m$ so that $a_{ij}^{(m)} > 0$.

In fact, if there exists a path from $i$ to $j$, there exists such a path with length less than $|V(G)|$. (if the length is greater than or equal to $|V(G)|$, then the path contains a cycle, which can be removed) So we can actually say a little more: $G$ is irreducible iff for all $i,j \in V(G)$, there exists $m < |V(G)|$ such that $a_{ij}^{(m)} > 0$.

This clearly is true iff every entry of $\sum_{m=1}^{V(G)} A^m$ has all positive entries.

2.3.4. $\Rightarrow$: Suppose that $X$ is an edge shift, i.e. there is a graph $G$ s.t. $X = X_G$. Then, for each $e, f \in E(G)$, clearly $f$ can follow $e$ in a walk iff $t(e) = i(f)$. And, for any such $e, f$, ef $\in L_2(X_G)$. (because we have the blanket assertion that $X$ is essential) So, $f \in X_G(e)$ iff $t(e) = i(f)$. But then for any $e, e' \in E(G)$, either $t(e) \neq t(e')$ and $X_G(e) \cap X_G(e') = \emptyset$, or $t(e) = t(e')$ and $X_G(e) = X_G(e')$.

$\Leftarrow$: Suppose that $X$ is a 1-step SFT with alphabet $A$, and the property that $X(a) = X(a')$ or $X(a) \cap X(a') = \emptyset$ for all $a, a' \in A$. Throw out any letters of $A$ which do not actually appear in points of $X$, and define a graph $G$ with vertex set $V(G) = \{X(a) : a \in A\}$. For each $b \in A$, by the assumed property of $X$, there is exactly one set $X(a)$ which contains $b$. (To be clear, there could be another letter $a' \in A$ so that $X(a')$ also contains $b$, but then $X(a) = X(a')$.)

Then, define an edge with name $b$ from $X(a)$ to $X(b)$. Create one such edge for each $b \in A$, and define $E(G)$ to consist of the set of edges thereby created. Note that each vertex has an incoming edge by definition. Each vertex also has an outgoing edge: for each $a \in A$, $X(a)$ is nonempty, and contains an element, call it $c$. Then by definition, there is an edge labeled $c$ from $X(a)$ to $X(c)$. So, $G$ is essential. We claim that $X = X_G$, which will show that $X$ is an edge shift. It suffices to show that $L(X) = L(X_G)$, and since $X_G$ and $X_G$ are both 1-step SFTs (by assumption and $X_G$ because all edge shifts are 1-step SFTs), it suffices to show that $L_2(X) = L_2(X_G)$.

$L_2(X) \subseteq L_2(X_G)$: Consider any $ab \in L_2(X)$. Then $b \in X(a)$, and so by definition of $G$, there is an edge $b$ from $X(a)$ to $X(b)$. By extendability of $L(X)$, there exists $c$ so that $ca \in L_2(X)$, and so $a \in X(c)$, and so there is an edge $a$ from $X(c)$ to $X(a)$. Then $t(a) = i(b)$, and $ab$ is a path in $G$, implying by essentiality of $G$ that $ab \in L_2(X_G)$.

$L_2(X_G) \subseteq L_2(X)$: Consider any $ab \in L_2(X_G)$. Then $ab$ is a path in $G$, and so $t(a) = i(b)$. By definition of $G$, $t(a) = X(a)$. Also by definition of $G$, since $i(b) = X(a)$, $b \in X(a)$, meaning that $ab \in L_2(X)$. 

2.4.7(b). Define $G_1$ to be the graph with adjacency matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, with vertices $I$ and $J$. If you perform an elementary out-splitting on $G_1$ by partitioning the outgoing edges from $I$, you will obtain exactly the graph $G_2$ with adjacency matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. See page 3 for a picture of the splitting.

2.4.7(d). Call $G_1$ and $G_2$ the graphs with the adjacency matrices $\begin{pmatrix} 2 \\ \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ respectively. If there were a sequence of splittings and amalgamations which could turn $G_1$ into $G_2$, then $X_{G_1}$ and $X_{G_2}$ would be conjugate, and would have to have the same number of points with least period $n$ for every $n$. This implies that $X_{G_1}$ and $X_{G_2}$ would have to have the same number of points with period $n$ for every $n$ as well, since you can count points of period $n$ by summing the numbers of points of least period $k$ for all $k$ dividing $n$. By Proposition 2.2.12(b), the traces of all powers of $\begin{pmatrix} 2 \\ \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ would then need to be equal, but this is not the case; for instance, $\text{Tr} \left( \begin{pmatrix} 2 \\ \end{pmatrix} \right)^2 = 4$, and $\text{Tr} \left( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right)^2 = 5$.

4.4.2(a). The communicating classes (or irreducible components) are $C_4 = \{2\}$, $C_3 = \{1, 5\}$, $C_2 = \{3, 6\}$, and $C_1 = \{4\}$. The reordered adjacency matrix is

$$
\begin{align*}
4 &\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 &\rightarrow & 0 & 0 & 3 & 0 & 0 & 0 \\ 6 &\rightarrow & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 &\rightarrow & 0 & 0 & 1 & 0 & 1 & 0 \\ 5 &\rightarrow & 2 & 0 & 0 & 1 & 1 & 0 \\ 3 &\rightarrow & 0 & 2 & 1 & 0 & 0 & 0 
\end{pmatrix}
\end{align*}
$$

See page 3 for a picture of the irreducible component graph.