4.4.5. We prove the contrapositive. Suppose that \( A \) is reducible. Then by the techniques of Section 4.4, the vertices of the associated graph \( G \) can be separated into irreducible components \( C_0, \ldots, C_k \) such that reordering the vertices of \( G \) into these components yields a matrix \( A' \) which is block diagonal. It is easy to check that the characteristic polynomial of \( A' \) is the same as that of \( A \); the easiest way to see this is to note that \( A' = PAP^{-1} \) for some permutation matrix \( P \). Then \( \det(A' - xI) = \det(PAP^{-1} - xI) = \det(P(A - xI)P^{-1}) = \det(P) \det(A - xI) \det(P^{-1}) = \det(A - xI) \).

Since \( A' \) is block diagonal, if the block matrices on its diagonal are \( A_1', \ldots, A_k' \), then \( \det(A' - xI) \) is block diagonal with block matrices \( A_1' - xI, \ldots, A_k' - xI \).

Therefore, \( \det(A' - xI) = \prod \det(A_i' - xI) \), which yields a nontrivial factorization of the characteristic polynomial of \( A' \). Since this is equal to the characteristic polynomial of \( A \), we are done.

4.5.3. Call \( G \) the graph with adjacency matrix \( A \), and suppose that \( P_0, \ldots, P_{p-1} \) are the period classes of \( G \). Fix any positive integer \( n \).

\[ \Rightarrow: \] Suppose that \( n \) and \( p \) share a common divisor \( k > 1 \). Then, if we denote by \( G^{[n]} \) the graph with adjacency matrix \( A^n \), clearly any edge in \( G^{[n]} \) beginning at a vertex in \( P_i \) terminates at a vertex in \( P_{i+n \mod p} \). Therefore, any path of length \( j \) in \( G^{[n]} \) beginning at a vertex in \( P_0 \) terminates at a vertex in \( P_{jn \mod p} \), which will always be a multiple of \( k \). There is then, for instance, no path in \( G^{[n]} \) from a vertex in \( P_0 \) to a vertex in \( P_1 \), implying that \( G^{[n]} \) is not irreducible, and also that \( A^n \) is not irreducible.

\[ \Leftarrow: \] Suppose that \( n \) and \( p \) are relatively prime, and again denote by \( G^{[n]} \) the graph with adjacency matrix \( A^n \). Since \( p = \text{per}(A) \), \( A^p \) is block diagonal with primitive matrices along the diagonal. Therefore, there exists \( M \) so that for any \( m > M \), \( A^{pm} \) is block diagonal with all of the block matrices strictly positive. In particular, all diagonal entries of \( A^{pm} \) are positive, so for any \( m > M \) and any \( i \in V(G) \), there exists a cycle starting and ending at \( i \) with length \( pm \). Now, consider any \( i, j \in V(G) \). By irreducibility, there exists a path \( \gamma \) in \( G \) from \( i \) to \( j \), say with length \( L \). By the previously mentioned fact, for any \( m > M \), there is a cycle of length \( pm \) starting and ending at \( i \). By inserting such cycles within \( \gamma \), we can get paths of length \( L + pm \) from \( i \) to \( j \) for any \( m > M \). But since \( p \) and \( n \) are relatively prime, there exists \( m' \) so that \( L + pm' \) is a multiple of \( n \). Therefore, there is a path in \( G^{[n]} \) from \( i \) to \( j \). Since \( i, j \) were arbitrary, \( G^{[n]} \) is irreducible, and so \( A^n \) is as well.

4.5.6. Denote by \( G \) the graph with adjacency matrix \( A \), and by \( G^{[p]} \) the graph with adjacency matrix \( A^p \). Denote by \( P_0, \ldots, P_{p-1} \) the period classes of \( G \). Clearly edges in \( G^{[p]} \) correspond to paths of length \( p \) in \( G \), and all edges in \( G^{[p]} \) start and end at vertices in the same period class. Denote by \( A_i \) the submatrix of \( A^p \) corresponding to edges in \( G^{[p]} \) starting and ending at vertices in \( P_i \).
Define the sliding block code where each \( \Phi \) starting and ending at vertices in \( P_i \), and the alphabet of \( X_{A_i} \) is the set of paths of length \( p \) in \( G \) starting and ending at vertices in \( P_{i+1} \). A point of \( X_A \) has the following form:

\[
x = (e_1 e_2 \cdots e_{p-1}) \Phi (e'_1 e'_2 \cdots e'_{p-1}) \cdots,
\]

where each \( (e_0^k e^k_1 \cdots e^{k}_{p-1}) \) is a path in \( G \) starting and ending at vertices in \( P_i \). Since mixing is invariant under conjugacy from \( X_A \) to \( X_{A_i+1} \).

Suppose that \( X \) is a mixing shift space, and that \( \phi \) is a conjugacy from \( X \) to another shift space \( Y \) (with memory \( m \) and anticipation \( n \)). Consider any words \( u, v \in L(Y) \). Since \( u \in L(Y) \), it is part of a biinfinite sequence \( y \in Y \), which has a preimage \( x \in X \). Therefore, there is a word \( u' \in L_{|u|+n+1}(X) \) such that \( \phi (u') = u \). Similarly, there exists \( v' \in L_{|v|+n+1}(X) \) such that \( \phi (v') = v \). By mixing of \( X \), there exists \( R \) so that for any \( r > R \), there is a word \( w' \) of length \( r \) for which \( u'w'v' \in L(X) \). Therefore, there exists \( x' \in X \) with \( u'w'v' \) as a subword. Then \( \phi (x') \in Y \), and contains a subword of the form \( u vw \) for some word \( w \) of length \( r + m + n \). Since \( u vw \) is a subword of \( \phi (x') \), it is in \( L(Y) \).

We have then shown that for any \( u, v \in L(Y) \), and \( r > R \), there exists a word \( w \) of length \( r + m + n \) such that \( u vw \in L(Y) \). Clearly this implies that for any \( s > R + m + n \), there exists \( w \) of length \( s \) for which \( u vw \in L(Y) \), and so we have shown that \( Y \) is mixing.

Consider any shift of finite type \( X \). Since mixing is invariant under conjugacy by (1), and since any shift of finite type is conjugate to an edge shift, we may assume that \( X \) is an edge shift \( X_G \) without loss of generality.

\[=\Rightarrow\]: Assume that an edge shift \( X_G \) is mixing, and denote by \( A \) the adjacency matrix of \( G \). For any \( i, j \) vertices in \( G \), choose edges \( e, f \) with \( t(e) = i \) and \( i(f) = j \). By mixing of \( X_G \), there exists \( N \) so that for any \( n > N \), there is a word \( w \) of length \( n \) for which \( ewf \in L(X_G) \), meaning that \( w \) is a path from \( i \) to
Take $N = \max_{i,j} N_{ij}$; then for any $i,j$ vertices of $G$ and $n > N$, there exists a path of length $n$ from $i$ to $j$, meaning that $A^n > 0$ for all $n > N$. Therefore, $A$ is primitive. Clearly this implies that $X_G$ is irreducible, and also that the greatest common divisor of cycle lengths in $G$ is equal to 1. Since there is a one-to-one correspondence between cycles of $G$ and periodic points of $X_G$, this also implies that the greatest common divisor of period lengths for $X_G = 1$.

$\Leftarrow$: Assume that $X_G$ is an edge shift which is irreducible and for which the greatest common divisor of period lengths is equal to 1. Then if $A$ is the adjacency matrix for $G$, clearly $A$ is irreducible, and since there is a one-to-one correspondence between cycles of $G$ and periodic points of $X_G$, the greatest common divisor of cycle lengths for $G$ is equal to 1. Therefore, by definition, $A$ is primitive. This means that there exists $N$ so that $A^n > 0$ for all $n > N$, or equivalently that for any $i,j$ vertices of $G$ and any $n > N$, there is a path from $i$ to $j$ of length $n$. But this implies mixing of $X_G$; for any $u,v \in L(X_G)$ and any $n > N$, there is a path $w$ from $t(u)$ to $i(v)$ in $G$ of length $n$, and so $uwv \in L(X_G)$.

4.5.16(a) $\Leftarrow$: Denote by $A$ the adjacency matrix for $G_X$, and by $G_X$ the minimal right-resolving presentation $(G_X, L_\infty)$ of $X$. Since $G_X$ is primitive, there exists $N$ so that $A^n$ is strictly positive for all $n > N$. This implies that for any $i,j \in V(G_X)$ and any $n > N$, there exists a path from $i$ to $j$ of length $n$ in $G_X$. Since $G_X$ is a labeled graph presenting $X$, for any words $u,v \in L(X)$, there exist paths $\gamma$ and $\delta$ in $G_X$ labeled by $u$ and $v$. Then for any $n > N$, there exists a path $\alpha$ from $t(\gamma)$ to $i(\delta)$ of length $n$ in $G_X$, meaning that $\gamma \alpha \delta$ is a path in $G_X$, and that its label $uwv$ is in $L(X)$, where $|w| = n$. This implies that $X$ is mixing.

$\Rightarrow$: Suppose that $X$ is mixing, and consider any vertices $i$ and $j$ in $V(G_X)$. Since $G_X$ is the minimal right-resolving presentation of $X$, it is irreducible and follower-separated. Therefore, there exist words $w_i$ and $w_j$ synchronizing to $i$ and $j$ respectively in $G_X$. By mixing of $X$, there exists $N_{ij}$ so that for any $n > N_{ij}$, there is a word $w$ of length $n$ for which $w_iww_j \in L(X)$. Since $G_X$ presents $X$, there is a path $\alpha$ in $G_X$ with labels $w_iww_j$. But, since $w_i$ synchronizes to $i$ and $w_j$ synchronizes to $j$, this implies that $\alpha$ has a subpath from $i$ to $j$ of length $n + |w_j|$. By taking $M_{ij} = N_{ij} + j$, we have shown that for any $m > M_{ij}$, there is a path from $i$ to $j$ in $G_X$ of length $m$. By taking $M$ to be the max of all $M_{ij}$, we see that for any $m > M$ and any $i,j \in V(G_X)$, there is a path of length $m$ from $i$ to $j$ in $G_X$, implying that $A^m$ is strictly positive for all $m > M$, which implies that $A$ is primitive.