1. Let \( P \) be the set of prime numbers, \( \{ z \in \mathbb{Z}^+ | \forall x \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+ (xy = z \rightarrow (x = z \lor x = 1)) \} \). Use set builder notation to specify the set of positive integers that are cubes of primes. (10 points)

2. For all \( n \) in the set of positive integers, define the set \( A_n \subset \mathbb{Z} \) by \( A_n = \{ z \in \mathbb{Z} | -n > z \lor n < z \} \). Express each of the following sets using bracket notation without any set operations. The universe is \( \mathbb{Z} \). (5 points each)

(a) \( \bigcup_{n=50}^{70} A_n \)

(b) \( \bigcap_{n=50}^{70} A_n \)

(c) \( A_{50} \triangle A_{70} \)

(d) \( \bigcup_{n=50}^{70} \overline{A_n} \)

3. Suppose you have access to partial information about the purchasing habits of people who are registered for a service called meTunes. Of 200 thousand people registered, 30 thousand bought music within a given
week, 15 thousand bought at least one audiobook, and 18 thousand bought at least one movie. Of these, 5 thousand bought music and at least one audiobook, 9 thousand bought music and at least one movie, and 6 thousand bought at least one movie and at least one audiobook. There were 2 thousand who made all three types of purchases. How many of the people registered purchased nothing that week? (10 points)

4. In the game of rencontre, an urn contains slips of paper numbered 1, 2, 3,...,n, well mixed. The player draws slips of paper one after another. The player wins if the first number drawn is a 1, or if the second number drawn is a 2, or if the third number drawn is a 3, etc. That is, the player wins if, in the sequence of numbers drawn, \( a_1, a_2, a_3, \ldots a_n \), there is a \( k \) with \( a_k = k \). (2 points each)

(a) Define a probability space with equally likely outcomes that can be used to model this process. Specify the sample space and the probability function for individual outcomes.

(b) Compute the probability that the player draws a 1 on the first draw.

(c) Set \( n=3 \). Compute the probability that the player wins.
(d) Set $n=3$. Suppose the player pays $1 to play, and gets $5 as a prize for winning (net = $4). What is the expected value of the game to the player?

(e) Consider the game with $n$ slips. The result of problem 8 shows that there are $n!(\sum_{i=0}^{n} \frac{(-1)^i}{i!})$ losing sequences the player can draw from $n$ slips. What is the probability that the player wins the game?

5. Consider a gamble in which the gambler buys a ticket for one dollar. The ticket pays $0 with probability .999, pays $40 with probability .0009, and $1000 with probability .0001.

(a) What is the net expected value of a ticket to the purchaser? (4 points)

(b) Suppose that the payouts of tickets purchased in different weeks are independent. What is the probability that a player will come out at least $38 ahead if the player purchases one ticket in week 1 and another ticket in week 2? (3 points)

(c) What is the expected value to the purchaser of 5 tickets each purchased in separate weeks? (3 points)

6. Colorado requires that infants be screened for metabolic diseases. The screen has a probability of approximately .027 of producing a false positive on a healthy infant. I don’t have data on the probability that it correctly produces a positive on an infant with one of the disorders for
which the tests screen. For the sake of argument, assume this probability is 1. In the Colorado newborn population, the probability that a randomly selected newborn will have one of the disorders checked in the screening is approximately .0014. (5 points each)

(a) What is the probability that a randomly selected newborn in Colorado had a positive screening test?

(b) What is the probability that a randomly selected newborn in Colorado has one of the screened disorders, given that infant’s test result was positive?

7. Suppose that a redundant system has three subsystems A, B, and C configured as follows: the system will function if subsystem A is available, or if B and C are available, i.e., $A \lor (B \land C)$. Suppose that the probability that subsystem A is available is $a$, the probability that B is available is $b$, the probability for C is $c$, and that these probabilities are mutually independent. What is the probability that the system is functioning? (10 points)

8. The goal of this problem is for you to verify a formula for the number of permutations $a_1, a_2, a_3, \ldots a_n$ of the numbers 1, 2, 3, ..., n with the property that, for all $k$ in 1, 2, 3, ..., n, $a_k \neq k$. (For example, for $n=3$, the permutation 2, 1, 3 has the property, but 3, 2, 1 does not.) But first we’ll talk about people and hats. Suppose you have $n$ people, $n>1$, each owning a hat. Collect all the hats and hand them out to the people, keeping track of who gets which hat. Denote by $d_n$ the numbers of ways there are to distribute the hats so that no hat is with its owner. Person 1 can get any of $n-1$ hats. In particular, say person 1 gets hat $k \neq 1$. If person $k$ happens to have hat 1, then the remaining $n-2$ people’s hats can be distributed in any of $d_{n-2}$ ways. If person $k$ does not have hat 1, there are $d_{n-1}$ ways for the remaining hats to be distributed: we can think of dropping person 1 and reassigning hat 1 to person $k$. This shows that $d_n = (n-1)(d_{n-1} + d_{n-2})$. From this formula, the values for $d_2$ and $d_3$, and the easily verified formula $n! = (n-1)((n-1)! + (n-2)!)$, prove
by induction that \( d_n = n! \left( \sum_{i=0}^{n} \left(\frac{-1}{i}\right) \right) \). Thinking of the people as the positions in the sequence and the hats as the \( a_i \)'s shows \( d_n \) is the solution to the permutation problem. (10 points)

9. For any positive integer \( n \), if \( A \) is any set containing \( n + 1 \) positive integers, all less than or equal to \( 2n \), then \( A \) contains two elements, \( b \) and \( c \), say, for which \( b \) is an integer multiple of \( c \). Prove this by induction. (10 points)