A Geometric Construction of Cyclic Cocycles on Twisted Convolution Algebras

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References

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(Advisor: Alexander Gorokhovsky)

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A simplicial manifold is a contravariant functor $M_\bullet : \Delta \to \text{Man}$. Objects $M_n$ and face maps $\delta_i^n$ for $n \geq 0$ and $0 \leq i \leq n$.

A simplicial differential $k$-form on a simplicial manifold $M_\bullet$ is a sequence of $k$-forms $\{\omega(n)\}$, with $\omega(n) \in \Omega^k(M_n \times \Delta^n)$, satisfying

$$(\text{id} \times \partial_i)^* \omega(n) = (\delta_i \times \text{id})^* \omega(n-1)$$

on $\Omega^k(M_n \times \Delta^{n-1})$ for $0 \leq i \leq n$ and $n \geq 1$. Here $\partial_i$ denotes the geometric face map $(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$.

$\Omega^k(M_\bullet)$ denotes simplicial differential $k$-forms on $M_\bullet$.

The complex $(\Omega^\bullet(M_\bullet), d)$ is the complex of compatible forms.

Let $\Omega^{r,s}(M_\bullet) = \coprod_{n \geq 0} \Omega^r(M_n) \otimes \Omega^s(\Delta^n)$. $(\Omega^{r,s}(M_\bullet), d_{dR}, d_\Delta)$ is the bicomplex of compatible forms, with $d_{dR}$ on $M_n$, $d_\Delta$ on $\Delta^n$. 

**Simplicial Notions (Dupont)**
Discrete Translation Groupoid

\( M \rtimes \Gamma \) denotes the translation groupoid where
- \( \Gamma \) a discrete group
- \( M \) a manifold
- \( M \rtimes \Gamma \to M \) an action \((x, g) \mapsto xg\)

The nerve \((M \rtimes \Gamma)_\bullet\) is a simplicial manifold with
- \((M \rtimes \Gamma)_{(k)} = M \times \Gamma^k\)
- \(\delta_i^k(m, g_1, \ldots, g_k) = \begin{cases} (mg_1, g_2, \ldots, g_k) & \text{if } i = 0 \\ (m, g_1, \ldots, g_ig_{i+1}, \ldots, g_k) & \text{for } 1 \leq i \leq k - 1 \\ (m, g_1, \ldots, g_{k-1}) & \text{if } i = k \end{cases}\)
- \(\sigma_j^k(m, g_1, \ldots, g_k) = (m, g_1, \ldots, g_j, 1_\Gamma, g_{j+1}, \ldots, g_k)\)
Gerbes on the Translation Groupoid

A **gerbe** \((L, \mu)\) over \(M \rtimes \Gamma\) consists of

- \(L \to M \rtimes \Gamma\) a line bundle over \((M \rtimes \Gamma)_{(1)} = M \times \Gamma\),

- \(\mu_{g_1, g_2} : L_{g_1} \otimes (L_{g_2})^{g_1} \sim L_{g_1 g_2}\) an isomorphism for every \(g_1, g_2 \in \Gamma\)

where \(L_g = L|_{M_g}\) on \(M_g = M \times \{g\}\) for each \(g \in \Gamma\)

satisfying the associativity condition that

\[
\begin{align*}
L_{g_1} \otimes (L_{g_2})^{g_1} \otimes (L_{g_3})^{g_1 g_2} 
&\xrightarrow{\mu_{g_1, g_2} \otimes \text{id}}
L_{g_1 g_2} \otimes (L_{g_3})^{g_1 g_2} \\
\downarrow \text{id} \otimes (\mu_{g_2, g_3})^{g_1} 
&
\downarrow \mu_{g_1 g_2, g_3} \\
L_{g_1} \otimes (L_{g_2 g_3})^{g_1} 
&\xrightarrow{\mu_{g_1, g_2 g_3}}
L_{g_1 g_2 g_3}
\end{align*}
\]

commutes for every \(g_1, g_2, g_3 \in \Gamma\).
Connections on Gerbes on the Translation Groupoid

Given a gerbe \((L, \mu)\) and \(\nabla\) a connection on \(L \to M \rtimes \Gamma\) let \(\nabla_g = \nabla|_{M_g}\) denote the connection on \(L_g\) for each \(g \in \Gamma\).

- For all \(g, h \in \Gamma\) define \(\alpha \in \Omega^1((M \rtimes \Gamma)_\otimes(2))\) the discrepancy of \(\nabla\) by
  \[
  \nabla_g \otimes 1 + 1 \otimes (\nabla_h)^g - \mu_{g_1, g_2}^* (\nabla_{gh}) = \alpha(g, h).
  \]

- Define \(\theta \in \Omega^2((M \rtimes \Gamma)_\otimes(1))\) by \([\theta_g, \cdot] = (\nabla_g)^2\) where \(\theta_g \in \Omega^2(M_g)\) for any \(g \in \Gamma\). The 2-forms \(\theta_g\) satisfy the condition
  \[
  \theta_g + \theta_h^g - \theta_{gh} = d\alpha(g, h)
  \]

- \((-\alpha, \theta) \in \Omega^1((M \rtimes \Gamma)_\otimes(2)) \oplus \Omega^2((M \rtimes \Gamma)_\otimes(1))\) is a degree 3 cocycle in \((\Omega^\bullet((M \rtimes \Gamma)_\otimes), d, \delta)\).
Twisted Convolution Algebra

- The convolution product on $C_c^\infty(M \rtimes \Gamma, L)$ is given by

$$ (f_1 * f_2)(x, g) = \sum_{g_1g_2 = g} f_1(x, g_1) \cdot f_2(xg_1, g_2) $$

for sections $f_1, f_2 \in C_c^\infty(M \times \Gamma, L)$, where $f_1(x, g_1) \cdot f_2(xg_1, g_2)$ is computed using the product $\mu_{g_1, g_2} : L_{g_1} \otimes (L_{g_2})^{g_1} \xrightarrow{\sim} L_g$.

- This gives $C_c^\infty(M \rtimes \Gamma, L)$ the structure of an algebra called the twisted convolution algebra.

- Our main goal is to construct periodic cyclic cocycles on this algebra.
Overview of Construction

Twisted Simplicial Complex $\Omega^*_\bullet(M\bullet)$ \xrightarrow{\text{JLO}} \text{Intermediate Complex} \xrightarrow{\text{Alg.}} (b, B)$-bicomplex of $C^\infty_c(M \rtimes \Gamma, L)$

Twisted by simplicial representative $\Theta_\bullet$ of Dixmier-Douady class

$\Gamma$-cochains valued in cochains on Matrix Algebras

We construct periodic cyclic cocycles here
Overview of Construction

Twisted Simplicial Complex $\Omega^*_-(M_\bullet)$ \(\xrightarrow{\text{JLO}}\) Intermediate Complex \(\xrightarrow{\text{Alg.}}\) \((b, B)\)-bicomplex of $C_\infty^c(M \rtimes \Gamma, L)$

- Twisted by simplicial representative $\Theta_\bullet$ of Dixmier-Douady class
- $\Gamma$-cochains valued in cochains on Matrix Algebras
- We construct periodic cyclic cocycles here
Direct Sum Bundle $\mathcal{E}$ and Matrix Algebras $\text{End } \mathcal{E}$

- $\mathcal{E} = \bigoplus_{g \in \Gamma} L_g$ defines an (infinite dimensional) bundle.
- $\nabla$ on $L$ defines direct sum connection $\nabla^\mathcal{E}$ on $\mathcal{E}$. The gerbe defines
  \[ \varphi_g : \bigoplus_{g' \in \Gamma} L_{g'} \sim \bigoplus_{g' \in \Gamma} \mu_{g,g'}^{-1}(L_{gg'}). \]

- Let $A \in \Omega^1(M \times \Gamma, \mathcal{E})$ denote the **discrepancy** of $\nabla^\mathcal{E}$ defined by
  \[ \nabla^\mathcal{E} - \varphi_g^* ( (\nabla^\mathcal{E})^g \otimes 1 + 1 \otimes \nabla_g) = -A(g) \in \Omega^1(M, \mathcal{E}). \]

- The bundle $\text{End } \mathcal{E} \to M$ is a bundle of finite rank endomorphisms.
- A section $f \in C^\infty(M, \text{End } \mathcal{E})$ may be decomposed into sections
  \[ E_{g_1,g_2}(f) \in C^\infty(M, \text{Hom}(L_{g_1}, L_{g_2})) \]

  and $C^\infty(M, \text{End } \mathcal{E})$ carries a $\Gamma$-action and matrix product.
Overview of Construction

Twisted Simplicial Complex $\Omega^* (M_\bullet)$ \[\xrightarrow{\text{JLO}}\] Intermediate Complex \[\xrightarrow{\text{Alg.}}\] $(b, B)$-bicomplex of $C^\infty (M \rtimes \Gamma, L)$

Twisted by simplicial representative $\Theta_\bullet$ of Dixmier-Douady class

$\Gamma$-cochains valued in cochains on Matrix Algebras

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Derivations $\nabla^k$

- $\mathcal{E}_k$ the pullback bundle of $\mathcal{E}$ over $(M \times \Gamma)_k = M \times \Gamma^k$ by $p_k : M \times \Gamma^k \rightarrow M$ with $p_k : (m, g_1, \ldots, g_k) \mapsto m$. Not simplicial.

- End $\mathcal{E}_k$ is simplicial. Denote by End $\mathcal{E}(k)$. $\pi_\bullet : \text{End } \mathcal{E}_\bullet \rightarrow (M \times \Gamma)_\bullet$.

- $\nabla^k : C^\infty((M \times \Gamma)_k \times \Delta^k, \mathcal{E}_k \times \Delta^k) \rightarrow \Omega^1((M \times \Gamma)_k \times \Delta^k, \mathcal{E}_k \times \Delta^k)$ is the derivation

\[
\nabla^k(g_1, \ldots, g_k)|_{(m,t_1,\ldots,t_k)} = (\nabla^\mathcal{E} + t_1 A(g_1) + \cdots + t_k A(g_1 \cdots g_k))|_{(m,t_1,\ldots,t_k)}
\]

with $0 \leq t_i \leq 1$, $\sum t_i \leq 1$.

- Extend to $\nabla^k : \Omega^\ell \rightarrow \Omega^{\ell+1}$ by Leibniz rule and to $\Omega^*(M, \text{End } \mathcal{E})$ by $\nabla^k \eta = [\nabla^k, p_k^* \eta]$ as usual, where $\eta \in \Omega^*(M, \text{End } \mathcal{E})$. 
Simplicial 2-form

The End $\mathcal{E}_k$-valued 2-forms $(\nabla^k)^2$ are not compatible.

**Theorem**

For each $k \geq 0$, define $\vartheta(k) \in \Omega^2((M \times \Gamma)_k \times \Delta^k, \text{End } \mathcal{E}_k \times \Delta^k)$ by

$$
\vartheta(k)(g_1, \ldots, g_k) = (\nabla^k(g_1, \ldots, g_k))^2 - \sum_{i=1}^k t_i \theta_{g_1 \cdots g_i} + \sum_{1 \leq i < j \leq k} \alpha(g_1 \cdots g_i, g_{i+1} \cdots g_j)(t_i dt_j - t_j dt_i).
$$

Then $\vartheta = \{\vartheta(k)\}$ is a compatible 2-form on $M_\bullet$ with values in $\text{End } \mathcal{E}_\bullet$, i.e. $\vartheta \in \Omega^2(M_\bullet, \text{End } \mathcal{E}_\bullet)$. We call $\vartheta$ the simplicial 2-form associated to $\nabla$.

The discrepancy $(\nabla^k)^2 - \vartheta(k)$ is a scalar valued form, $(\nabla^k)^2(a) = [\vartheta(k), a]$. 
Simplicial Dixmier-Douady Form

Theorem

Let $\Theta_{(k)} = \nabla^k \vartheta_{(k)}$. The collection of 3-forms $\Theta_{(k)}$ define a simplicial 3-form $\{\Theta_{(k)}\} \in \Omega^3(M_\bullet)$. Furthermore, $\Theta_{(k)}^{0,3} = 0$ as an element of $\Omega^{r,s}(M_\bullet)$.

We can give an explicit formula for $\Theta$

$$\Theta_{(k)}(g_1, \ldots, g_k) = \sum_{i=1}^{k} \theta_{g_1 \cdots g_i} dt_i + \sum_{1 \leq i < j \leq k} d\alpha(g_1 \cdots g_i, g_{i+1} \cdots g_j)(t_i dt_j - t_j dt_i) + 2\alpha(g_1 \cdots g_i, g_{i+1} \cdots g_j) dt_i dt_j,$$

Quasi-isomorphism of integration along simplices maps $\Theta$ to $(-\alpha, \theta)$. 
Overview of Construction

Twisted Simplicial Complex $\Omega^\ast_\bullet (M\bullet)$

Twisted by simplicial representative $\Theta_\bullet$ of Dixmier-Douady class

JLO $\rightarrow$ Intermediate Complex

$\Gamma$-cochains valued in cochains on Matrix Algebras

Alg. $\rightarrow$ $(b, B)$-bicomplex of $C^\infty_c(M \rtimes \Gamma, L)$

We construct periodic cyclic cocycles here
Twisted Simplicial Cohomology

- $\Omega^{r,s}_{-}(M_{\bullet})[u]$ – rescaled version of bicomplex of compatible forms with $u$ a formal variable of degree +2.

- $\Omega^{k}_{-}(M_{\bullet})[u] = \bigoplus_{r+s=k} \Omega^{r,s}_{-}(M_{\bullet})[u]$ is the total complex.

- Rescale the following for use on $\Omega^{r,s}_{-}(M_{\bullet})[u]$:
  - $ud_{dR} : \Omega^{r,s}_{-}(M_{\bullet})[u] \longrightarrow \Omega^{r+1,s}_{-}(M_{\bullet})[u]$ the exterior differential on $M_{n}$
  - $d_{\Delta} : \Omega^{r,s}_{-}(M_{\bullet})[u] \longrightarrow \Omega^{r,s+1}_{-}(M_{\bullet})[u]$ the exterior differential on $\Delta^{n}$
  - $\nabla^{k} \Rightarrow \nabla_{u}^{k}$
  - $\vartheta \Rightarrow \vartheta_{u}$
  - $\Theta \Rightarrow \Theta_{u}$ and $\Theta_{u} \wedge \cdot : \Omega^{k}_{-}(M_{\bullet})[u] \rightarrow \Omega^{k+1}_{-}(M_{\bullet})[u]$

- Define the complex of $\Theta$-twisted complex of compatible forms to be $(\Omega^{*}_{-}(M_{\bullet})[u], ud_{dR} + d_{\Delta} - \Theta_{u} \wedge \cdot)$ as long as $\Theta^{0,3} = 0$. 
Overview of Construction

Twisted Simplicial Complex $\Omega^*_\cdot(M\cdot)$ \xrightarrow{JLO} Intermediate Complex \xrightarrow{Alg.} $(b, B)$-bicomplex of $C^\infty_c(M \rtimes \Gamma, L)$

Twisted by simplicial representative $\Theta\cdot$ of Dixmier-Douady class

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We construct periodic cyclic cocycles here

E. Angel (CU) Twisted Convolution Algebras June 14, 2010 17 / 22
Morphisms

JLO Morphism

Theorem

Let \( C^n(\Gamma, K) \) denote the space of inhomogenous \( n \)-cochains of \( \Gamma \) with values in \( K \) with coboundary \( \delta_\Gamma \), \( b \) the Hochschild coboundary, and \( B \) the Connes coboundary. There is a morphism

\[
\tau_\nabla : (\Omega^*(((M \rtimes \Gamma)_\bullet)[u], ud_{dR} + d\Delta - \Theta u \wedge \cdot) \rightarrow (C^\bullet(\Gamma, \overline{CC}^\bullet(C_c^\infty(M, \text{End} \mathcal{E}))[u^{-1}, u]), b + uB + \delta_\Gamma)
\]

where, for \( \omega = \{\omega_{(k)}\} \in \Omega^*_-(M_\bullet), a_0, \ldots, a_n \in C_c^\infty(M, \text{End} \mathcal{E}), \)

\[
\tau_\nabla(\omega)(\tilde{a}_0, a_1, \ldots, a_n) =
\sum_k \int_M \int_{\Delta^k} \omega_{(k)} \wedge \left( \int_{\Delta^n} \text{tr} \left( \tilde{a}_0 e^{-\sigma_0(\vartheta u)_{(k)}} \nabla_u^k(a_1)e^{-\sigma_1(\vartheta u)_{(k)}} \cdots \nabla_u^k(a_n)e^{-\sigma_n(\vartheta u)_{(k)}} \right) d\sigma_1 \cdots d\sigma_n \right)
\]
Overview of Construction

Twisted Simplicial Complex $\Omega^*_\cdot(M_\cdot)$ \xrightarrow{\text{JLO}} \text{Intermediate Complex} \xrightarrow{\text{Alg.}} (b, B)$-bicomplex of $C_c^\infty(M \rtimes \Gamma, L)$

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First Algebraic Morphism

For $\tilde{C}^* (\Gamma, K)$ homogenous $n$-cochains of $\Gamma$ with values in $K$ with coboundary $\tilde{\delta}_\Gamma$ there is an (explicit) contracting homotopy

$$h : \tilde{C}^{k+1} (\Gamma, \overline{CC}^n (C^\infty_c (M, \text{End} \mathcal{E}))) \rightarrow \tilde{C}^k (\Gamma, \overline{CC}^n (C^\infty_c (M, \text{End} \mathcal{E})))$$

that satisfies $(\tilde{\delta}_\Gamma h + h \tilde{\delta}_\Gamma) = \text{id}$.

Theorem

There is an (explicit) morphism

$$\psi_1 : (\tilde{C}^* (\Gamma, \overline{CC}^* (C^\infty_c (M, \text{End} \mathcal{E}))[u^{-1}, u]), \tilde{\delta}^G + b + uB)$$

$$\rightarrow (\overline{CC}^* (C^\infty_c (M, \text{End} \mathcal{E}))[u^{-1}, u], b + uB)^\Gamma$$
Second Algebraic Morphism

**Theorem**

*There is a morphism*

\[
\Psi_2 : \overline{CC}^\bullet(C_c^\infty(M, \text{End } \mathcal{E})[u^{-1}, u], b + uB)^\Gamma \rightarrow \overline{CC}^\bullet(C_c^\infty(M \rtimes \Gamma, L)[u^{-1}, u], b + uB)
\]

*defined by the formula*

\[
\Psi_2(c)(\tilde{a}_{g_0}, \ldots, a_{g_n}) = c(E_{1,g_0}(\tilde{a}_{g_0}), \ldots, E_{g_0 \cdots g_{i-1}, g_0 \cdots g_i}(a_{g_{i-1}}^{g_0 \cdots g_{i-1}}), \ldots, E_{g_0 \cdots g_{n-1}, 1}(a_{g_{n-1}}^{g_0 \cdots g_{n-1}}))
\]

*for any \(g_0, \ldots g_n \in \Gamma\) such that \(g_0 \cdots g_n = 1\) and \(\Psi_2(c) = 0\) otherwise. Here \(c\) is a cochain in \(C^n(C_c^\infty(M, \text{End } \mathcal{E}))^\Gamma\) and \(a_{g_i} \in C_c^\infty(M_{g_i}, L_{g_i})\) for \(0 \leq i \leq n\) with \(\tilde{a}_{g_0} = (a_{g_0}, \lambda)\) for \(\lambda \in \mathbb{C}\).
**Conclusion**

Twisted Simplicial Complex $\Omega^*_-(M\dot{\cdot})$ \xrightarrow{JLO} Intermediate Complex $\xrightarrow{\psi_2 \circ \psi_1}$ $(b, B)$-bicomplex of $C_c^\infty(M \rtimes \Gamma, L)$

Putting together these morphisms

**Theorem**

The map

$$\psi_2 \circ \psi_1 \circ \tau_{\nabla} : (\Omega^*_-( (M \rtimes \Gamma)\dot{\cdot})[u], ud_{dR} + d\Delta - \Theta_u \wedge \cdot)$$

$$\longrightarrow \overline{CC}^\bullet (C_c^\infty (M \rtimes \Gamma, L)[u^{-1}, u], b + uB)$$

is a morphism.