Algebraic Cuntz-Pimsner rings
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Cuntz-Pimsner $C^*$-algebras

- Cuntz-Pimsner $C^*$-algebras was introduced in 1995 by Pimsner as a generalization of $C^*$-crossed products by $\mathbb{Z}$ and Cuntz-Krieger algebras.
- In 2004 Katsura generalized the definition of Cuntz-Pimsner algebras to accommodate non-injective left actions. This makes it possible to construct graph $C^*$-algebras.
- Other classes of $C^*$-algebras that can be constructed as Cuntz-Pimsner algebras include:
  - $C^*$-algebras of topological graphs
  - $C^*$-algebras associated to subshifts (also called shift spaces)
  - $C^*$-algebras of labelled graphs
  - $C^*$-algebras of self similar groups
Properties of Cuntz-Pimsner algebras

- There are 6-exact sequences which can be used to compute the $K$- and the $KK$-theory of Cuntz-Pimsner algebras.
- Sufficient and necessary conditions for when a Cuntz-Pimsner algebra is nuclear are know.
- Sufficient and necessary conditions for when a Cuntz-Pimsner algebra is exact are know.
- Sufficient conditions for when a Cuntz-Pimsner algebra satisfies the Universal Coefficient Theorem of Rosenberg and Schochet are know.
Several of the classes of $C^*$-algebras that can be constructed as Cuntz-Pimsner algebras have algebraic analogs:

- The algebraic crossed product of a single ring automorphism.
- The fractional skew monoid ring of a single corner automorphism.
- Leavitt path algebras.

It is therefore natural to construct algebraic analogs of Cuntz-Pimsner algebras.
Covariant representations of $R$-systems

Definition of an $R$-system

Let $R$ be a ring. An $R$-system is a triple $(P, Q, \psi)$ where $P$ and $Q$ are $R$-bimodules, and $\psi$ is an $R$-bimodule homomorphism from $P \otimes Q$ to $R$.

Definition of a covariant representation

Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. We say that a quadruple $(T, S, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$ on $B$ if

1. $B$ is a ring,
2. $S : P \to B$ and $T : Q \to B$ are additive maps,
3. $\sigma : R \to B$ is a ring homomorphism,
4. $S(pr) = S(p)\sigma(r)$, $S(rp) = \sigma(r)S(p)$, $T(qr) = T(q)\sigma(r)$ and $T(rq) = \sigma(r)T(q)$ for $r \in R$, $p \in P$ and $q \in Q$,
5. $\sigma(\psi(p \otimes q)) = S(p)T(q)$ for $p \in P$, $q \in Q$. 

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The Toeplitz representation

**Theorem**

Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. Then there exists a **universal** covariant representation $(\iota_R, \iota_Q, \iota_P, T_{(P, Q, \psi)})$ of $(P, Q, \psi)$.

That $(\iota_R, \iota_Q, \iota_P, T_{(P, Q, \psi)})$ is a **universal** covariant representation means that if $(T, S, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$, then there exists a unique ring homomorphism $\psi_{(T, S, \sigma, B)} : T_{(P, Q, \psi)} \rightarrow B$ such that:

- $\psi_{(T, S, \sigma, B)} \circ \iota_R = \sigma$,
- $\psi_{(T, S, \sigma, B)} \circ \iota_Q = T$,
- $\psi_{(T, S, \sigma, B)} \circ \iota_P = S$. 

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An example

- Let $R$ be a ring with local units.
- Let $\varphi \in \text{Aut}(R)$ be a ring automorphism.
- Let $P := R_\varphi$ be the $R$-bimodule with the right action defined by $p \cdot r = p\varphi(r)$ and the left action defined by $r \cdot p = rp$.
- Let $Q := R_{\varphi^{-1}}$ be the $R$-bimodule with the right action defined by $q \cdot r = q\varphi^{-1}(r)$ and the left action defined by $r \cdot q = rq$.
- $\psi_\varphi : R_\varphi \otimes R_{\varphi^{-1}} \rightarrow R$, $p \otimes q \mapsto p\varphi(q)$ is an $R$-bimodule homomorphism.
- Thus $(R_\varphi, R_{\varphi^{-1}}, \psi_\varphi)$ is an $R$-system.
The Toeplitz ring of \((R_\varphi, R_{\varphi^{-1}}, \psi_\varphi)\)

- Let \((S, T, \sigma, B)\) be a covariant representation of \((R_\varphi, R_{\varphi^{-1}}, \psi_\varphi)\).
  If we for \(r \in R\) and \(n \in \mathbb{N}\) let \([r, n] = S^n(r), [r, -n] = T^n(r)\) and \([r, 0] = \sigma(r)\), then
  \[ [r_1, n] + [r_2, n] = [r_1 + r_2, n] \]
  \[ [r_1, n_1][r_2, n_2] = [r_1\varphi^{n_1}(r_2), n_1 + n_2] \] if \(n_1\) and \(n_2\) both are non-positive, or both are non-negative, or if \(n_1\) is non-negative and \(n_2\) is non-positive.

- If on the other hand \(B\) is a ring with elements \(\{[r, n] \mid r \in R, n \in \mathbb{Z}\}\) satisfying (1) and (2) above and we define \(\sigma : R \to B, S : R_\varphi \to B\) and \(T : R_{\varphi^{-1}} \to B\) by
  \[ \sigma(r) = [r, 0] \]
  \[ S(r) = [r, 1] \]
  \[ T(r) = [r, -1] \]
  then \((S, T, \sigma, B)\) is a covariant representation of \((R_\varphi, R_{\varphi^{-1}}, \psi_\varphi)\).
The Toeplitz ring of \((R_\varphi, R_{\varphi^{-1}}, \psi_\varphi)\)

Thus \(T(R_\varphi, R_{\varphi^{-1}}, \psi_\varphi)\) is the universal ring generated by elements
\([r, n] \mid r \in R, \ n \in \mathbb{Z}\) satisfying

1. \([r_1, n] + [r_2, n] = [r_1 + r_2, n]\)

2. \([r_1, n_1][r_2, n_2] = [r_1 \varphi^{n_1}(r_2), n_1 + n_2]\) if \(n_1\) and \(n_2\) both are non-positive, or both are non-negative, or if \(n_1\) is non-negative and \(n_2\) is non-positive,
Injective, surjective and graded representations

A covariant representation \((S, T, \sigma, B)\) of an \(R\)-system is said to be

- **injective** if \(\sigma\) is injective,
- **surjective** if \(B\) is generated by \(\sigma(R) \cup S(P) \cup T(Q)\),
- **graded** if there exists a \(\mathbb{Z}\)-grading \(\bigoplus_{n \in \mathbb{Z}} B^{(n)}\) of \(B\) such that \(\sigma(R) \subseteq B^{(0)}, S(P) \subseteq B^{(-1)}\) and \(T(Q) \subseteq B^{(1)}\).

**Remark**

The Toeplitz representation \((\iota_R, \iota_Q, \iota_P, T(P, Q, \psi))\) is injective, surjective and graded.

We will try to classify, up to isomorphism, all injective, surjective, graded representations of a given \(R\)-system.
Adjointable operators

Definition of adjointable operators

Let \((P, Q, \psi)\) be an \(R\)-system. A right \(R\)-module homomorphism \(U : Q \to Q\) is said to be adjointable with respect to \(\psi\) if there exists a left \(R\)-module homomorphism \(V : P \to P\) such that 
\[
\psi(p \otimes T(q)) = \psi(S(p) \otimes q)
\]
for all \(p \in P\).

The set \(\mathcal{L}_\psi(Q)\) of adjointable operators on \(Q\) is then an \(R\)-algebra.

\(\mathcal{L}_\psi(P)\) can be defined in a similar way.
Finite rank operators

For \( q \in Q \) and \( p \in P \) we let \( \theta_{q,p} : Q \to Q \) be the map
\[ q' \mapsto q\psi(p \otimes q'), \]
and we let \( \theta_{p,q} : P \to P \) be the map \( p' \mapsto \psi(p' \otimes q)p \).
\( \theta_{p,q} \) is an adjoint of \( \theta_{q,p} \) (and \( \theta_{q,p} \) is the adjoint of \( \theta_{p,q} \)).
Thus \( \theta_{q,p} \in \mathcal{L}_\psi(Q) \) and \( \theta_{p,q} \in \mathcal{L}_\psi(P) \).
Let \( \mathcal{F}_\psi(Q) = \text{span}\{\theta_{q,p} | q \in Q, \ p \in P\} \) and \( \mathcal{F}_\psi(P) = \text{span}\{\theta_{p,q} | q \in Q, \ p \in P\} \).
\( \mathcal{F}_\psi(Q) \) is then a two-sided ideal in \( \mathcal{L}_\psi(Q) \) and \( \mathcal{F}_\psi(P) \) is a two-sided ideal in \( \mathcal{L}_\psi(P) \).
Condition (FS)

**Definition**

An $R$-system $(P, Q, \psi)$ is said to satisfy condition (FS) if for all finite sets $\{q_1, \ldots, q_n\} \subseteq Q$ and $\{p_1, \ldots, p_m\} \subseteq P$ there exist $\Theta \in \mathcal{F}_\psi(Q)$ and $\Delta \in \mathcal{F}_\psi(P)$ such that $\Theta(q_i) = q_i$ and $\Delta(p_j) = p_j$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$ respectively.
**An example of condition (FS)**

- Let $R$ be a unital ring.
- Let $Q$ be an $R$-bimodule which is finitely generated and projective as a right $R$-module.
- Let $P = Q^* = \text{Hom}_R(Q, R)$.
- Then $P$ is finitely generated and projective as a left $R$-module and $Q = P^*$.
- Define $\psi : P \otimes Q \to R$ by $p \otimes q \mapsto p(q)$.
- Then $(P, Q, \psi)$ is an $R$-system.
- It follows from the Dual Basis Lemma that there exist $q_1, \ldots, q_n \in Q$ and $p_1, p_2, \ldots, p_n \in P$ such that $\sum_{i=1}^n q_i p_i(q) = q$ for every $q \in Q$.
- Dually, there exist $p'_1, \ldots, p'_m \in P$ and $q'_1, \ldots, q'_n \in Q$ such that $\sum_{j=1}^m q'_j(p)p'_j = p$ for all $p \in P$.
- Thus $(P, Q, \psi)$ satisfies condition (FS).
Representations of $R$-systems which satisfies condition (FS)

We will now classify, up to isomorphism, all injective, surjective, graded covariant representations of a given $R$-system which satisfies condition (FS).

**Proposition**

If $R$ is a ring, $(P, Q, \psi)$ is an $R$-system which satisfies condition (FS) and $(T, S, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$, then there exists a unique ring homomorphism $\pi_{T, S}: \mathcal{F}_\psi(Q) \to B$ satisfying $\pi_{T, S}(\theta_{q, p}) = T(q)S(p)$ for $q \in Q$ and $p \in P$. 
Cuntz-Pimsner invariant representations

- If \((P, Q, \psi)\) is an \(R\)-system, then we define for each \(r \in R\) an operator on \(\Delta(r) : Q \rightarrow Q\) by \(\Delta(r)(q) = rq\).
- A two-sided ideal \(J\) in \(R\) is called \(\psi\)-compatible if \(J \subseteq \Delta^{-1}(\mathcal{F}_\psi(Q))\).

**Definition**

Let \(R\) be a ring, let \((P, Q, \psi)\) be an \(R\)-system satisfying condition \((FS)\) and let \(J\) be a \(\psi\)-compatible two-sided ideal of \(R\). A covariant representation \((T, S, \sigma, B)\) of \((P, Q, \psi)\) is said to be *Cuntz-Pimsner invariant relative to \(J\)* if \(\pi_{T, S}(\Delta(x)) = \sigma(x)\) for every \(x \in J\).
Relative Cuntz-Pimsner representations

- Let \((P, Q, \psi)\) be an \(R\)-system satisfying condition (FS) and let \(J\) be a \(\psi\)-compatible two-sided ideal of \(R\).
- We let \(\mathcal{T}(J)\) be the two-sided ideal of \(R\) generated by \(\{\iota_R(r) - \pi_{\iota_Q, \iota_P}(\Delta(r)) \mid r \in J\}\).
- Let \(O_{(P,Q,\psi)}(J) = \mathcal{T}_{(P,Q,\psi)}/\mathcal{T}(J)\) and \(i^J_R = \rho_J \circ \iota_R\), \(i^J_Q = \rho_J \circ \iota_Q\) and \(i^J_P = \rho_J \circ \iota_P\) where \(\rho_J : \mathcal{T}_{(P,Q,\psi)} \to O_{(P,Q,\psi)}(J)\) is the quotient map.

**Theorem**

\((\iota^J_P, \iota^J_Q, \iota^J_R, O_{(P,Q,\psi)}(J))\) is a universal \(J\)-Cuntz-Pimsner invariant covariant representation of \((P, Q, \psi)\).
Moreover, \((\iota^J_P, \iota^J_Q, \iota^J_R, O_{(P,Q,\psi)}(J))\) is surjective and graded, and it is injective if and only if \(\ker \Delta \cap J = \{0\}\).

We say that a two-sided ideal \(J\) of \(R\) is **faithful** if \(\ker \Delta \cap J = \{0\}\).
For a covariant representation \((S, T, \sigma, B)\) let
\[J(S, T, \sigma, B) = \{ x \in R \mid \sigma(r) \in \pi_{T,S}(\mathcal{F}_\psi(Q)) \}.\]

- \(J(S, T, \sigma, B)\) is a \(\psi\)-compatible two-sided ideal of \(R\).
- \(J(S, T, \sigma, B)\) is faithful if and only if \((S, T, \sigma, B)\) is injective.
- If \(J\) is a \(\psi\)-compatible two-sided ideal of \(R\), then \(J \subseteq J(S, T, \sigma, B)\) if and only if \((S, T, \sigma, B)\) is Cuntz-Pimsner invariant relative to \(J\).
Classification of covariant representations

Theorem
Let $J$ be a faithful, $\psi$-compatible two-sided ideal of $R$ and let $(S, T, \sigma, B)$ be a covariant representation.

1. If there exists a ring homomorphism $\eta : O_{(P,Q,\psi)}(J) \rightarrow B$ such that $\eta \circ \iota^J_Q = T$, $\eta \circ \iota^J_P = S$ and $\eta \circ \iota^J_R = \sigma$, then the representation $(S, T, \sigma, B)$ is Cuntz-Pimsner invariant with respect to $J$.

2. If the representation $(S, T, \sigma, B)$ is Cuntz-Pimsner invariant with respect to $J$, then there exists a unique ring homomorphism $\eta^J_{(S,T,\sigma,B)} : O_{(P,Q,\psi)}(J) \rightarrow B$ such that $\eta^J_{(S,T,\sigma,B)} \circ \iota^J_Q = T$, $\eta^J_{(S,T,\sigma,B)} \circ \iota^J_P = S$ and $\eta^J_{(S,T,\sigma,B)} \circ \iota^J_R = \sigma$.  

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Classification of covariant representations

Theorem

Let $J$ be a faithful, $\psi$-compatible two-sided ideal of $R$ and let $(S, T, \sigma, B)$ be a covariant representation. If the representation $(S, T, \sigma, B)$ is Cuntz-Pimsner invariant with respect to $J$, then the ring homomorphism $\eta(S, T, \sigma, B)$ is an isomorphism if and only if $(S, T, \sigma, B)$ is surjective, injective and graded and $J = J(S, T, \sigma, B)$. 

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Classification of covariant representations

It follows that

- Every surjective, injective and graded covariant representation of \((P, Q, \psi)\) is isomorphic to \((\iota_J^P, \iota_J^Q, \iota_J^R, O_{(P,Q,\psi)}(J))\) for some faithful \(\psi\)-compatible two-sided ideal \(J\) of \(R\).

- If \(J_1\) and \(J_2\) are two faithful \(\psi\)-compatible two-sided ideals of \(R\), then there exists a ring homomorphism \(\phi\) from \(O_{(P,Q,\psi)}(J_1)\) to \(O_{(P,Q,\psi)}(J_2)\) satisfying \(\phi \circ \iota_{J_1}^Q = \iota_{J_2}^Q, \phi \circ \iota_{J_1}^P = \iota_{J_2}^P\) and \(\phi \circ \iota_{J_1}^R = \iota_{J_2}^R\) if and only if \(J_1 \subseteq J_2\).
A faithful, $\psi$-compatible two-sided ideal $J$ of $R$ is said to be uniquely maximal if $J' \subseteq J$ for any faithful, $\psi$-compatible two-sided ideal $J'$ of $R$.

**Definition**

If there exists a uniquely maximal faithful, $\psi$-compatible two-sided ideal $J$ of $R$, then we define the *Cuntz-Pimsner* representation of $(P, Q, \psi)$ to be the representation $(\iota_P^{CP}, \iota_Q^{CP}, \iota_R^{CP}, \mathcal{O}_{(P,Q,\psi)}(J)) = (\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J))$. 

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An example without a uniquely maximal faithful, $\psi$-compatible ideal

- Let $R$ be the group $\mathbb{Z}^3$.
- We make $R$ into a ring by defining a multiplication on $R$ by
  \[(x, y, z)(x', y', z') = (xx', xy' + yx', xz' + zx').\]
- $(1, 0, 0)$ is then a unit for $R$.
- Define $\delta : R \to R$ to be the map $(x, y, z) \mapsto (x, y - z, 0)$.
- Then $\delta$ is a homomorphism.
- Let $P = Q = \{(x, y, 0) | x, y \in \mathbb{Z}\} \subseteq R$.
- We make $P$ and $Q$ into $R$-bimodules by $p \cdot r := p\delta(r)$, $r \cdot p := \delta(r)p$,
  $q \cdot r := q\delta(r)$, $r \cdot q := \delta(q)p$.
- Define $\psi : P \otimes Q \to R$ by $\psi(p \otimes q) = pq$.
- Then $(P, Q, \psi)$ is an $R$-system which satisfies condition (FS).
An example without a uniquely maximal faithful, \( \psi \)-compatible ideal

Let \( J_1 = \{(0, y, 0) \mid y \in \mathbb{Z}\} \) and \( J_2 = \{(0, 0, z) \mid z \in \mathbb{Z}\} \).

Then \( J_1 \) and \( J_2 \) are both faithful, \( \psi \)-compatible two-sided ideal of \( R \) and they are both maximal in the sense that \( J_i \subseteq J \) implies \( J_i = J \) for all faithful, \( \psi \)-compatible two-sided ideals \( J \) of \( R \).

Since \( J_1 \not= J_2 \), this implies that there cannot be a uniquely maximal faithful, \( \psi \)-compatible two-sided ideal of \( R \).

Remark

In the \( C^* \)-algebraic case the analog of \( J = \Delta^{-1}(\mathcal{F}_\psi(Q)) \cap (\ker \Delta)^\perp = \{x \in \Delta^{-1}(\mathcal{F}_\psi(Q)) \mid xy = yx = 0 \forall y \in \ker \Delta\} \) is always a uniquely maximal faithful, \( \psi \)-compatible.
The crossed product by a single automorphism

- Let $R$ be a ring with local units.
- Let $\varphi \in \text{Aut}(R)$ be a ring automorphism.
- Let $P =: R_\varphi$ be the $R$-bimodule with the right action defined by $p \cdot r = p\varphi(r)$ and the left action defined by $r \cdot p = rp$.
- Let $Q := R_{\varphi^{-1}}$ be the $R$-bimodule with the right action defined by $q \cdot r = q\varphi^{-1}(r)$ and the left action defined by $r \cdot q = rq$.
- $\psi \varphi : R_\varphi \otimes R_{\varphi^{-1}} \to R$, $p \otimes q \mapsto p\varphi(q)$ is an $R$-bimodule homomorphism.
- The $R$-system $(R_\varphi, R_{\varphi^{-1}}, \psi \varphi)$ satisfies condition (FS).
- $\Delta^{-1}(\mathcal{F}_\psi(Q)) = R$ and $\ker \Delta = \{0\}$.
- $R$ is a uniquely maximal faithful, $\psi$-compatible two-sided ideal of $R$, so $\mathcal{O}(R_\varphi, R_{\varphi^{-1}}, \psi \varphi)$ is defined and it equal to $\mathcal{O}(R_\varphi, R_{\varphi^{-1}}, \psi \varphi)(R)$. 

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The crossed product by a single automorphism

- If \((S, T, \sigma, B)\) is a covariant representation of \((R_\varphi, R_\varphi^{-1}, \psi_\varphi)\) and we for \(r \in R\) and \(n \in \mathbb{N}\) let \([r, n] = S^n(r)\), \([r, -n] = T^n(r)\) and \([r, 0] = \sigma(r)\), then
  1. \([r_1, n] + [r_2, n] = [r_1 + r_2, n]\)
  2. \([r_1, n_1][r_2, n_2] = [r_1 \varphi^{n_1}(r_2), n_1 + n_2]\) if \(n_1\) and \(n_2\) both are non-positive, or both are non-negative, or if \(n_1\) is non-negative and \(n_2\) is non-positive.

- \((S, T, \sigma, B)\) is Cuntz-Pimsner invariant relative to \(R\) if and only if \([r_1, n_1][r_2, n_2] = [r_1 \varphi^{n_1}(r_2), n_1 + n_2]\) for \(n_1 < 0\) and \(n_2 > 0\).

- Thus \(\mathcal{O}_{(R_\varphi, R_\varphi^{-1}, \psi_\varphi)}\) is the universal ring generated by elements \([r, n] \mid r \in R, \ n \in \mathbb{Z}\) satisfying
  1. \([r_1, n] + [r_2, n] = [r_1 + r_2, n]\),
  2. \([r_1, n_1][r_2, n_2] = [r_1 \varphi^{n_1}(r_2), n_1 + n_2]\).

- I.e., \(\mathcal{O}_{(R_\varphi, R_\varphi^{-1}, \psi_\varphi)}\) is isomorphic to the crossed product of \((R, \varphi)\).
The graded uniqueness theorem

We say that a relative Cuntz-Pimsner ring $\mathcal{O}_{(P,Q,\psi)}(J)$ satisfies the *graded uniqueness theorem* if the following holds:

1. If $B$ is a $\mathbb{Z}$-graded ring and $\eta : \mathcal{O}_{(P,Q,\psi)}(J) \to B$ is a graded ring homomorphism such that $\eta \circ \iota^J_R$ is injective, then $\eta$ is injective.

**Theorem**

$\mathcal{O}_{(P,Q,\psi)}(J)$ satisfies the graded uniqueness theorem if and only if $J$ is a maximal faithful, $\psi$-compatible two-sided ideal of $R$ in the sense that $J \subseteq J'$ implies $J' = J$ for all faithful, $\psi$-compatible two-sided ideals $J'$ of $R$.

**Corollary**

$\mathcal{O}_{(P,Q,\psi)}$ is defined, then it satisfies the graded uniqueness theorem.
\(\psi\)-invariants ideals

Definition of a \(\psi\)-invariant ideal

A two-sided ideal \(I\) of \(R\) is called \(\psi\)-invariant if \(\psi(p \otimes xq) \in I\) for \(x \in I\), \(p \in P\) and \(q \in Q\).

If \(I\) is a \(\psi\)-invariant two-sided ideal of \(R\), then we define:

- \(QI := \text{span}\{qx : q \in Q, x \in I\}\),
- \(IP := \text{span}\{xp : p \in P, x \in I\}\),
- \(R_I := R/I\), \(Q_I := Q/QI\) and \(P_I := P/IP\),
- \(\psi_I : IP \otimes QI \to R_I\) by \(\psi_I(q_I(p) \otimes q_I(q)) = q_I(\psi(p \otimes q))\) for \(p \in P\) and \(q \in Q\).

Then \((IP, Q_I, \psi_I)\) is an \(R_I\)-system. If \((P, Q, \psi)\) satisfies condition \((FS)\), then so does \((IP, Q_I, \psi_I)\).
Graded ideals of $\mathcal{O}_{(P,Q,\psi)}(J)$

**Theorem**

The set of all the graded two-sided ideals of $\mathcal{O}_{(P,Q,\psi)}(J)$ corresponds bijectively to the set of all pairs $(I, K)$ of ideals of $R$ satisfying

- $I \subseteq K$,
- $I$ is $\psi$-invariant,
- $q_I(K) \subseteq \Delta_I^{-1}(\mathcal{F}_\psi(Q_I))$ and $q_I(K) \cap \ker \Delta_I = \{0\}$,
- $J \subseteq K$. 
Crossed products by a single automorphism

Let us return to our example where $R$ is ring with local units. and $\varphi \in \text{Aut}(R)$ is a ring automorphism.

- A two-sided ideal $I$ of $R$ is $\psi$-invariant if and only if $\varphi(I) \subseteq I$.
- If $I$ is a $\psi$-invariant ideal of $R$, then $q_I(R) \subseteq \Delta_I^{-1}(\mathcal{F}_{\psi_I}(Q_I))$ and $q_I(R) \cap \ker \Delta_I = \{0\}$ if and only if $\varphi(I) = I$.
- Thus we have a bijective correspondence between two-sided graded ideals of $O_{(R_\varphi, R_{\varphi^{-1}} \psi_\varphi)}$ and two-sided ideals $I$ of $R$ satisfying $\varphi(I) = I$. 
Summary

- We have for all $R$-systems $(P, Q, \psi)$ constructed the Toeplitz ring $T_{(P,Q,\psi)}$.
- We have introduced the condition (FS) and for $R$-systems $(P, Q, \psi)$ satisfying condition (FS) and faithful, $\psi$-compatible two-sided ideals $J$ of $R$ constructed the relative Cuntz-Pimsner ring $O_{(P,Q,\psi)}(J)$.
- If there exists a uniquely maximal faithful, $\psi$-compatible two-sided ideal $J$ of $R$, then we define the Cuntz-Pimsner ring $O_{(P,Q,\psi)}$ of $(P, Q, \psi)$ to be $O_{(P,Q,\psi)}(J)$.
- The crossed product of a single ring automorphism, Leavitt path algebras and the fractional skew monoid ring of a corner automorphism can be constructed as Cuntz-Pimsner rings.
- $O_{(P,Q,\psi)}$ satisfies the graded uniqueness theorem.
- We have classified all graded two-sided ideals of $O_{(P,Q,\psi)}$. 

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