Group actions on topological graphs

Valentin Deaconu, Alex Kumjian, John Quigg

(work in progress)

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We recall basic facts about topological graphs and their $C^*$-algebras, with examples.

We introduce the Cayley graph of a finitely generated locally compact group and the skew product $E \times_c G$ of a topological graph $E$ by a group $G$ via a cocycle $c : E^1 \to G$.

We define the action of a group $G$ on a topological graph $E$. We give a structure theorem for free and proper actions, and define the quotient graph $E/G$.

We also introduce the fundamental group and the universal covering of a topological graph via a geometric realization. We give examples, one having the Baumslag-Solitar group as fundamental group.
Let $E = (E^0, E^1, s, r)$ be a topological graph. Recall that $E^0$ (vertices) and $E^1$ (edges) are locally compact (Hausdorff) spaces, $s, r : E^1 \to E^0$ are continuous maps, and $s$ is a local homeomorphism.

The $C^*$-algebra $C^*(E)$ is the Cuntz-Pimsner algebra $\mathcal{O}_\mathcal{H}$ of the $C^*$-correspondence $\mathcal{H} = \mathcal{H}(E)$ over $A = C_0(E^0)$, obtained as a completion of $C_c(E^1)$ using

$$\langle \xi, \eta \rangle (v) = \sum_{s(e) = v} \overline{\xi(e)} \eta(e), \xi, \eta \in C_c(E^1)$$

$$(\xi \cdot f)(e) = \xi(e)f(s(e)), (f \cdot \xi)(e) = f(r(e))\xi(e).$$
Examples

- **Example 1.** Let $E^0 = E^1 = \mathbb{T}$, $s(z) = z$, and $r(z) = e^{2\pi i \theta} z$ for $\theta \in [0, 1]$ irrational. Then $C^*(E) \cong A_\theta$, the irrational rotation algebra.

- **Example 2.** Let $E^0 = E^1 = X$, for $X$ a locally compact metric space, let $s = id$ and let $r = h : X \to X$ be a homeomorphism. Then $C^*(E) \cong C_0(X) \rtimes \mathbb{Z}$, since $C^*(E)$ is the universal $C^*$-algebra generated by $C_0(X)$ and a unitary $u$ satisfying $\hat{h}(f) = u^*fu$ for $f \in C_0(X)$, where $\hat{h}(f) = f \circ h$.

- **Example 3.** Let $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$. Take

$$E^0 = E^1 = \mathbb{T}, s(z) = z^n, r(z) = z^m.$$  

If $m \notin n\mathbb{Z}$, then $C^*(E)$ is simple and purely infinite.
The Cayley graph

- The Cayley graph of a finitely generated locally compact group. A locally compact group $G$ is finitely generated if there is a finite subset $S \subset G$ such that $G = \langle S \rangle$. If $S = \{h_1, h_2, \ldots, h_n\}$, define the Cayley graph $E = E(G, S)$ with $E^0 = G$, $E^1 = S \times G$, $s(h, g) = g$, and $r(h, g) = gh$.

- Then $E(G, S)$ becomes a topological graph. For $G$ discrete finitely generated, we get the usual notion of Cayley graph. (The Cayley graph may change if we change the set of generators).

- Example. For $G = (\mathbb{R}, +)$ and $S = \{1, \theta\}$, where $\theta < 0$ is irrational, the Cayley graph $E$ has $E^0 = \mathbb{R}$, $E^1 = \{1, \theta\} \times \mathbb{R}$ and $s(1, x) = x$, $r(1, x) = x + 1$, $s(\theta, x) = x$, $r(\theta, x) = x + \theta$.

- Then $C^*(E)$ is simple and purely infinite, isomorphic to $\mathcal{O}_2 \rtimes_\alpha \mathbb{R}$. Here $\alpha_t(V_0) = e^{it\theta}V_0$, $\alpha_t(V_1) = e^{it}V_1$ for $t \in \mathbb{R}$ and $V_0, V_1$ are the standard generators of the Cuntz algebra $\mathcal{O}_2$. 

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Skew products

- **Skew products of topological graphs.** Let $E = (E^0, E^1, s, r)$ be a topological graph, let $G$ be a locally compact group, and let $c : E^1 \rightarrow G$ be continuous.

- Define the skew product graph $E \times_c G = (E^0 \times G, E^1 \times G, \tilde{s}, \tilde{r})$, where

  $$\tilde{s}(e, g) = (s(e), g), \quad \tilde{r}(e, g) = (r(e), gc(e)).$$

- Then $E \times_c G$ becomes a topological graph using the product topology. If $E$ has one vertex and $n$ loops $\{e_1, \ldots, e_n\}$ and if $G$ has a set of generators $S = \{h_1, \ldots, h_n\}$ such that $c(e_i) = h_i, i = 1, \ldots, n$ then we get the Cayley graph $E(G, S)$. 
Graph morphisms

Let $E, F$ be two topological graphs. A graph morphism $\phi : E \to F$ is a pair of continuous maps $\phi = (\phi^0, \phi^1)$ such that the diagram

\[
\begin{array}{cccc}
E^0 & \xleftarrow{s} & E^1 & \xrightarrow{r} & E^0 \\
\downarrow{\phi^0} & & \downarrow{\phi^1} & & \downarrow{\phi^0} \\
F^0 & \xleftarrow{s} & F^1 & \xrightarrow{r} & F^0
\end{array}
\]

is commutative.

A graph morphism $\phi$ is a graph covering if both $\phi^0, \phi^1$ are covering maps.

An isomorphism is a graph morphism $\phi = (\phi^0, \phi^1)$ such that $\phi^i$ are homeomorphisms for $i = 0, 1$. It follows that $\phi^{-1} = ((\phi^0)^{-1}, (\phi^1)^{-1})$ is also a graph morphism.
A locally compact group $G$ acts on $E$ if there are continuous maps $\lambda^i : G \times E^i \to E^i$ for $i = 0, 1$ such that $g \mapsto \lambda_g$ is a homomorphism from $G$ into $Aut(E)$.

The action $\lambda$ is called **free** if $\lambda^0_g(v) = v$ for some $v \in E^0$ implies $g = 1_G$. In this case the action of $G$ is also free on $E^1$.

The action is called **proper** if the maps $G \times E^0 \to E^0 \times E^0$, $(g, v) \mapsto (\lambda^0_g(v), v)$ and $G \times E^1 \to E^1 \times E^1$, $(g, e) \mapsto (\lambda^1_g(e), e)$ are proper. (It is sufficient to require properness of the first map).

**Example.** A group $G$ acts freely and properly on a skew product $E \times_c G$ by $\lambda^0_g(v, h) = (v, gh)$ and $\lambda^1_g(e, h) = (e, gh)$. In particular, a f.g. group $G$ acts freely and properly on its Cayley graph $E = E(G, S)$. The quotient graph $E/G$ has $|S|$ loops and one vertex.
Principal $G$-bundles and the quotient graph

A map $q : P \to X$ is called a **principal $G$-bundle** if there is a free and proper action of $G$ on $P$ such that $P/G$ can be identified with $X$.

**Theorem**

Given $F = (F^0, F^1, s, r)$ a topological graph, a principal $G$-bundle $P \to F^0$ and an isomorphism of pull-backs $s^*(P) \cong r^*(P)$, there is a topological graph $E = (E^0, E^1, \tilde{s}, \tilde{r})$ with a free and proper action of $G$ such that $E^0 = P, E^1 = s^*(P)$ and $F \cong E/G$. Moreover, every topological graph $E$ on which $G$ acts freely and properly arises this way.

**Corollary**

The topological graph $E$ constructed above is $G$-equivariantly isomorphic to a skew product $F \times_c G$ iff the principal bundle $E^0 \to F^0$ is trivial.
The geometric realization of a topological graph $E$ is

$$R(E) := E^1 \times [0, 1] \sqcup E^0 / \sim,$$

where $(e, 0) \sim s(e)$ and $(e, 1) \sim r(e)$ (a kind of double mapping torus).

If the group $G$ acts on the topological graph $E$, then $G$ acts on $R(E)$ by

$$g \cdot (e, t) = (\lambda_g^1(e), t), \quad e \in E^1, t \in [0, 1], \quad g \cdot v = \lambda_g^0(v), \quad v \in E^0.$$

The fundamental group $\pi_1(E)$ is by definition $\pi_1(R(E))$. The universal covering $\tilde{E}$ of $E$ is a simply connected graph which covers $E$.

The group $\pi_1(E)$ acts freely on $\tilde{E}$, and the quotient graph $\tilde{E}/G$ is isomorphic to $E$.

If $E$ is discrete, then $\pi_1(E)$ is free, and the universal covering is a tree $T$. 
Proposition.

Let $E$ be such that $R = R(E)$ has a universal covering space $\tilde{R}$. Then $\tilde{R}$ is homeomorphic to the geometric realization of a simply connected topological graph $\tilde{E}$, which covers $E$.

Proof.

Let $\rho : \tilde{R} \to R$ be the canonical map, let $\tilde{E}^0 = \rho^{-1}(E^0)$, and let $\tilde{E}^1 = \rho^{-1}(E^1 \times \{1/2\})$. In order to define the range and the source maps, we use the unique path lifting property of the map $\rho$.

Corollary

To obtain other coverings of a graph $E$ as above, we take a subgroup $H$ of $\pi_1(E)$, and take the corresponding topological graph of the quotient space $\tilde{R}/H$. 
Examples

Example 1. Let $E$ with $E^0 = E^1 = \mathbb{T}$ and with $s(z) = z$, $r(z) = e^{2\pi i \theta} z$ for $\theta$ irrational. Then $R(E)$ is homeomorphic to $\mathbb{T}^2$, hence $\pi_1(E) \cong \mathbb{Z}^2$.

The universal covering $\tilde{E}$ has $\tilde{E}^0 = \tilde{E}^1 = \mathbb{R} \times \mathbb{Z}$, and $s(y, k) = (y, k)$, $r(y, k) = (y + \theta, k + 1)$. Here $\mathbb{Z}^2$ acts on $\tilde{E}$ by $(j, m) \cdot (y, k) = (y + m\theta + j, k + m)$, and $\tilde{E}/\mathbb{Z}^2 \cong E$.

Example 2. Let $h : X \to X$ be a homeomorphism, and let $E$ with $E^0 = E^1 = X$, $s = id$ and $r = h$. Then $R(E)$ is homeomorphic to the mapping torus of $h$.

The universal covering $\tilde{E}$ has $\tilde{E}^0 = \tilde{E}^1 = \tilde{X} \times \mathbb{Z}$, where $\tilde{X}$ is the universal covering of $X$. The source and range maps are $s(y, k) = (y, k)$, $r(y, k) = (\tilde{h}(y), k + 1)$, where $\tilde{h} : \tilde{X} \to \tilde{X}$ is a lifting of $h$.

We have $\pi_1(E) \cong \pi_1(X) \rtimes \mathbb{Z}$, and the action of $\pi_1(X) \rtimes \mathbb{Z}$ on $\tilde{X} \times \mathbb{Z}$ is given by $(g, m) \cdot (y, k) = (g \cdot \tilde{h}^m(y), k + m)$. 
Examples cont’d

Example 3. Let again $E^0 = E^1 = \mathbb{T}$ with $s(z) = z^p$, $r(z) = z^q$ for $p, q$ positive integers. Then $R(E)$ is obtained from a cylinder, where the two boundary circles are identified using the maps $s$ and $r$.

Figure: The case $p = 2$, $q = 3$. 
Examples cont’d

- Then $\pi_1(E)$ is isomorphic to the Baumslag-Solitar group $B(p, q) = \langle a, b \mid ab^p a^{-1} = b^q \rangle$.

- For $p = 1$ or $q = 1$, this group is a semi-direct product and it is amenable. For $p \neq 1, q \neq 1$ and $(p, q) = 1$, it is not amenable.

- The universal covering space of $R(E)$ is obtained from the Cayley graph of $B(p, q)$ by filling out the squares. It is the cartesian product $T \times \mathbb{R}$, where $T$ is the Bass-Serre tree of $B(p, q)$, viewed as an HNN-extension of $\pi_1(\mathbb{T})$.

- Recall that $B(p, q)$ is the quotient of the free product $\pi_1(\mathbb{T}) \ast \mathbb{Z}$ by the relation $as_\ast(b)a^{-1} = r_\ast(b)$, where $a$ is the generator of $\mathbb{Z}$, $b \in \pi_1(\mathbb{T})$, and $s_\ast, r_\ast : \pi_1(\mathbb{T}) \hookrightarrow \pi_1(\mathbb{T})$. 
Examples cont’d

\[ \begin{align*}
ba^{-1} & \; \rightarrow \; b \; \rightarrow \; b^{-1} & \; \rightarrow \; b^2 & \; \rightarrow \; b^{-1}b^3 & \; \rightarrow \; b^3 & \; \rightarrow \; b^{-1}b^2 & \; \rightarrow \; b^2a \; \rightarrow \; b^2a^{-1}b & \; \rightarrow \; b^2a b & \; \rightarrow \; b^2a b^{-1}b^2 & \; \rightarrow \; b^2a b^{-1}b^3 & \; \rightarrow \; b^2a^{-1}b^2 & \; \rightarrow \; b^2a^{-1}b^3 & \; \rightarrow \; b^5
\end{align*} \]

\[ \begin{align*}
a & \; \rightarrow \; ab & \; \rightarrow \; ab^2 = b^3a & \; \rightarrow \; bab & \; \rightarrow \; bab^2 = b^4a & \; \rightarrow \; b^2ab & \; \rightarrow \; b^2ab^2 = b^5a
\end{align*} \]

**Figure:** Cayley complex for \( B(2, 3) \).
Examples cont’d

- The 1-skeleton is the directed Cayley graph of $B(2, 3)$, where the generators $a, b$ multiply on the right. The group action is given by left multiplication.

- In the corresponding tree $T$, each vertex has 5 edges. The vertex set $T^0$ is identified with the left cosets $g\langle b \rangle \in B(2, 3)/\langle b \rangle$, and the edge set $T^1$ with the left cosets $g\langle b^2 \rangle \in B(2, 3)/\langle b^2 \rangle$.

- The source and range maps are given by
  
  $s(g\langle b^2 \rangle) = g\langle b \rangle, \quad r(g\langle b^2 \rangle) = ga^{-1}\langle b \rangle$ for $g \in B(2, 3)$.

- We have $\tilde{E}^0 \cong T^0 \times \mathbb{R}, \tilde{E}^1 \cong T^1 \times \mathbb{R}$ with
  
  $\tilde{s}(t, y) = (s(t), py), \tilde{r}(t, y) = (r(t), qy)$ for $t \in T^1$ and $y \in \mathbb{R}$.

- The group $B(p, q)$ acts freely and properly on $\tilde{E}$, and the quotient graph $\tilde{E}/B(p, q)$ is isomorphic to $E$.

- In particular, $\tilde{E}$ is not a skew product. We have $C^*(\tilde{E}) \rtimes_r B(p, q)$ strongly Morita equivalent to $C^*(E)$. 
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