THE RELATIVE WEAK ASYMPTOTIC
HOMOMORPHISM PROPERTY FOR INCLUSIONS
OF FINITE von NEUMANN ALGEBRAS

(joint work with Junsheng Fang, Mingchu Gao)
BACKGROUND

Unitary normalizers were studied by Dixmier for $A \subseteq M$:

$\mathcal{U}(M) = \{\text{unitaries in } M\}$

$\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$.

For maximal abelian self-adjoint subalgebras (masas) $A$,

(i) $A$ is singular if $\mathcal{N}_M(A)^\prime\prime = A$

(ii) $A$ is Cartan if $\mathcal{N}_M(A)^\prime\prime = M$. 
CRITERION FOR SINGULARITY

Definition 1. $B \subseteq M$ has the weak asymptotic homomorphism property (WAHP) if there are unitaries $\{u_n \in B\}$ so that

$$\lim_{n \to \infty} \| \mathbb{E}_B(x u_n y) - \mathbb{E}_B(x) u_n \mathbb{E}_B(y) \|_2 = 0$$

for $x, y \in M$. □

In the case of masas we have

Theorem 2. $A \subseteq M$ has the WAHP if and only if $A$ is singular.

(⇒) is Robertson, Sinclair, S.

(⇐) is Sinclair, S., White, Wiggins.
SKETCH OF (⇒)

If $v$ is a unitary normalizer of $B$ then $vu_nv^* \in B$ so
\[ E_B(vu_nv^*) = vu_nv^*. \]
Since $\|vu_nv^*\|_2 = 1$, we get from
\[ \lim_{n \to \infty} \|E_B(vu_nv^*) - E_B(v)u_nE_B(v^*)\|_2 = 0 \]
that $\|E_B(v)\|_2 = 1$ so
\[ v = E_B(v) \in B. \]

For a given sequence $\{u_n\}$, it is enough to verify
\[ \lim_{n \to \infty} \|E_B(xu_ny) - E_B(x)u_nE_B(y)\|_2 = 0 \]
for $x, y$ in a total subset of $M$ in the WOT.
EXAMPLE

\[ M = L(\mathbb{F}_2), \ A = \langle a \rangle'' . \]

Take \( u_n = a^n \).

For any words \( v, w \in \mathbb{F}_2 \setminus \langle a \rangle \),

\[ \mathbb{E}_A(v) = \mathbb{E}_A(w) = 0, \]

while \( \mathbb{E}_A(va^n w) = 0 \) for \( n \) large enough. Thus \( A \) has the WAHP.
CHIFAN’S GENERALIZATION

**Definition 3.** A triple $B \subseteq N \subseteq M$ has the relative weak asymptotic homomorphism property (RWAHP) if there are unitaries $\{u_n \in B\}$ so that

$$\lim_{n \to \infty} \|\mathbb{E}_B(xu_ny) - \mathbb{E}_B(\mathbb{E}_N(x)u_n\mathbb{E}_N(y))\|_2 = 0.$$ 

The WAHP for $B \subseteq M$ is equivalent to the RWAHP for the triple

$$B \subseteq B \subseteq M.$$ 

**Theorem 4** (Chifan). *If $B$ is a masa then*

$$B \subseteq \mathcal{N}_M(B)^{''} \subseteq M$$

*has the RWAHP.*
QUASI–NORMALIZERS

**Definition 5.** For an inclusion $B \subseteq M$, $x \in M$ is a quasi-normalizer if there exist $x_1, \ldots, x_n \in M$ with

$$Bx \subseteq \sum_{i=1}^{n} x_i B, \quad xB \subseteq \sum_{i=1}^{n} Bx_i,$$

$x \in M$ is a one sided quasi-normalizer if the first containment holds. We write $qN_M(B)$ and $W^*(qN^{(1)}_M(B))$ for the algebras generated by the quasi-normalizers and the one sided version.

Normalizers are one sided quasi-normalizers but not the other way around.
EXAMPLE

$S_n$ permutation group on $n$ letters.

$$B = Q times S_2 \subseteq Q \rtimes S_3 = M.$$  

$$g = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} \in S_3.$$  

Then

$$Bg \subseteq gB + g^{-1}B$$

and you cannot do better than this. Thus $g$ is a one sided quasi-normalizer but not a normalizer.
CHARACTERIZATION OF RWAHP

**Theorem 6** (Fang, Gao, S). *The following are equivalent:

(i) The triple $B \subseteq N \subseteq M$ has the RWAHP;

(ii) $W^*(q\mathcal{N}_M^{(1)}(B)) \subseteq N$. 
CONSEQUENCES

Corollary 7. The triple

\[ B \subseteq W^*(qN^{(1)}_M(B)) \subseteq M \]

has the RWAHP.

Corollary 8. \( B \subseteq M \) has the WAHP if and only if

\[ W^*(qN^{(1)}_M(B)) = B. \]

Corollary 9. If \( B \subseteq M \) is a masa then

\[ W^*(qN^{(1)}_M(B)) = N_M(B)'' . \]

Corollary 10. If \( B \subseteq M \) is a masa then \( B \) is singular if and only if

\[ qN^{(1)}_M(B) \subseteq B. \]
GROUP-SUBGROUP INCLUSIONS

$H \subseteq G$ induces $L(H) \subseteq L(G)$. At the group level, one sided quasi-normalizers $g$ are defined by the requirement that there exist $g_1, \ldots, g_n \in G$ such that

$$Hg \subseteq \bigcup_{i=1}^{n} g_i H.$$ 

Let

$\Gamma = \{\text{one sided quasi-normalizers}\}$,

$H_1 = \Gamma \cap \Gamma^{-1}$ (maximal subgroup of $\Gamma$),

$H_2 = \text{group generated by } \Gamma$.

**Theorem 11.** Let $H \subseteq G$ and let $x = \sum \alpha_g g$ be a one sided quasi-normalizer of $L(H)$. For each $g_0$ so that $\alpha_{g_0} \neq 0$, we have $g_0 \in \Gamma$. 
Corollary 12. Let $H \subseteq G$ be an inclusion of groups.

(i) $q\mathcal{N}_{L(G)}(L(H))'' = L(H_1);

(ii) $W^*(q\mathcal{N}^{(1)}_{L(G)}(L(H))) = L(H_2)$.

Example 13. $\mathbb{Z}$ acts on $\mathbb{F}_\infty$ by $g_i \mapsto g_{i+1}$.

$G = \mathbb{F}_\infty \rtimes \mathbb{Z}$, $H$ is generated by $g_0, g_1, g_2, \ldots$.

If $t$ is the generator of $\mathbb{Z}$ then

$$t^{-1} \in H_2 \quad \text{but} \quad t^{-1} \notin H_1.$$
GROUP CONDITIONS

(C1) For each \( g \in G \setminus H \), \( \{hgh^{-1} : h \in H\} \) is infinite;

(C2) Given \( g_1, \ldots, g_n \in G \setminus H \) there exists \( h \in H \) with

\[
g_i h g_j \not\in H, \quad 1 \leq i, j \leq n;
\]

(C3) If \( g \in G \) and there exists \( \{g_1, \ldots, g_n\} \subseteq G \) such that

\[
H g \subseteq \bigcup_{i=1}^{n} g_i H,
\]

then \( g \in H \). (Same as \( \Gamma = H \).)

When \( H \) is abelian, (C1) is equivalent to \( L(H) \) maximal abelian, and (C2) and (C3) are each equivalent to singularity of \( L(H) \) in \( L(G) \).
NORMALIZERS

\[ \mathcal{N}_G(H) = \{ g \in G : gHg^{-1} = H \}. \]

**Theorem 14.** Let \( H \subseteq G \) with \( L(H) \) maximal abelian in \( L(G) \). Then

\[ \mathcal{N}_{L(G)}(L(H))'' = L(\mathcal{N}_G(H)). \]

In particular, \( L(H) \) is singular if and only if \( \mathcal{N}_G(H) = H \), and is Cartan precisely when \( H \) is a normal subgroup of \( G \). \( \square \)

The characterization of the RWAHP and Chifan’s theorem tell us that

\[ \mathcal{N}_{L(G)}(L(H))'' = W^*(\Gamma). \]

We have (trivially) that \( \mathcal{N}_G(H) \subseteq \Gamma \), and the reverse containment is handled by
Lemma 15. Let $H$ be an abelian subgroup of $G$ such that each $g \in G \setminus H$ has infinitely many $H$-conjugates ($L(H)$ is a masa in $L(G)$). If $g \in G$ satisfies

$$Hg \subseteq \bigcup_{i=1}^{n} g_i H$$

for some $g_1, \ldots, g_n \in G$, then $g$ is a normalizer of $H$ inside $G$. 