This continues the discussion of how to solve systems of the form \( \frac{dY}{dt} = AY \), where \( A \) is a 2 \times 2 matrix. In this handout, we consider the case where \( A \) is a 2 \times 2 matrix with complex eigenvalues \( \lambda = \alpha \pm i\beta \), where \( \alpha \) and \( \beta \) are real and \( \beta \neq 0 \). Note that we may always take \( \lambda = \alpha \pm i\beta \) where \( \beta > 0 \).

Follow the following steps.

1. Pick either \( \lambda = \alpha + i\beta \) or \( \lambda = \alpha - i\beta \) and stick with it! For purposes of illustration, assume we pick \( \lambda = \alpha + i\beta \).

2. For this choice of eigenvalue \( \lambda \), find an eigenvector \( V_c \) (with complex entries) corresponding to \( \lambda \).

3. Write
   \[
   Y_c(t) = e^{\lambda t} V_c = e^{(\alpha+i\beta)t} V_c = e^{\alpha t} (\cos (\beta t) + i \sin (\beta t)) (V_{re} + i V_{im})
   \]
   where both \( V_{re} \) and \( V_{im} \) have real entries. Continuing the calculation, write
   \[
   Y_c(t) = Y_{re}(t) + i Y_{im}(t)
   \]
   where \( Y_{re}(t) \) and \( Y_{im}(t) \) have real entries.

4. Then \( Y_{re}(t) \) and \( Y_{im}(t) \) are linearly independent real solutions of the system \( \frac{dY}{dt} = AY \). The general solution of this system is \( Y(t) = k_1 Y_{re}(t) + k_2 Y_{im}(t) \) where \( k_1 \) and \( k_2 \) are arbitrary constants.

**Example:** \( \frac{dY}{dt} = AY \) where \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). The numbers below correspond to the numbered steps above.

1. The eigenvalues are solutions of
   \[
   0 = \det (A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 + 1
   \]
   i.e., \( \lambda = \pm i \). So take \( \lambda = i \).

2. An complex eigenvector \( V_c \) corresponding to \( \lambda = i \) is computed as follows.
   \[
   \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -ix + y \\ -y - ix \end{pmatrix}
   \]
   so \( y = ix \). Letting \( x = 1 \), we get \( y = i \) and so \( V_c = \begin{pmatrix} 1 \\ i \end{pmatrix} \).
3. A complex solution of this system is

\[ Y_c(t) = e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} = (\cos(t) + i \sin(t)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + i \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} \]

4. So, \( Y_{re}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \) and \( Y_{im}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \) are two linearly independent real solutions of the system. The general solution of this system is

\[ Y(t) = k_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + k_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \]

where \( k_1 \) and \( k_2 \) are arbitrary constants.

**Problem:**

Solve the initial value problem

\[ \frac{dY}{dt} = BY, \quad Y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

where \( B = \begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix} \). Note that there is only a sign difference between the matrix \( B \) and the matrix \( A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \) which was assigned in the handout solving-systems-1.pdf and which has the title “Solving Linear Systems – Real, Distinct Eigenvalues.”