Admissible Orders of Jordan Loops

Michael Kinyon, Kyle Pula, & Petr Vojtěchovský

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A commutative loop or quasigroup is **Jordan** if it satisfies the following identity:

\[(x^2y)x = x^2(yx)\]

Goodaire & Keeping show that the loops rings of Jordan loops have certain “nice” properties.

**Question**

*For which orders do non-associative Jordan loops exist?*
Main Result

Theorem (Goodaire & Keeping)

NJLs exist for all even orders and for all orders $n \equiv 0 \mod 7$.

Theorem (Kinyon, Pula, & Vojtěchovský)

NJLs exist for all orders $n \geq 6$ and $n \neq 9$. 
Main Technique: Amalgam Construction

Background:

- Construction originally appears in Bruck
- Name “Amalgam” comes from Foguel

Ingredients:

- \((G, \cdot)\) ... an idempotent quasigroup
- \((L, \ast)\) ... a loop
- \((Q, \bullet)\) ... a quasigroup with \(Q := L \setminus \{1\}\)
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Build a multiplication on the set \((G \times Q) \cup \{1\}\) as follows:

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- Place copies of \((Q,\bullet)\) in the off diagonal blocks.
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Example of Amalgam Construction

\[
\begin{array}{c|ccc}
(G, \cdot) & f & g & h \\
\hline
f & f & h & g \\
g & h & g & f \\
h & g & f & h \\
\end{array}
\quad
\begin{array}{c|ccc}
(L, \ast) & 1 & a & b \\
\hline
1 & 1 & a & b \\
a & a & b & 1 \\
b & b & 1 & a \\
\end{array}
\quad
\begin{array}{c|cc}
(Q, \circ) & a & b \\
\hline
a & a & b \\
b & b & a \\
\end{array}
\]

\begin{array}{c|cccccccc}
& 1 & (f, a) & (f, b) & (g, a) & (g, b) & (h, a) & (h, b) \\
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1 & 1 & (f, a) & (f, b) & (g, a) & (g, b) & (h, a) & (h, b) \\
(f, a) & (f, a) & (f, b) & (f, 1) & (h, a) & (h, b) & (g, a) & (g, b) \\
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\end{array}
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We write \( A(G, L, Q) \) for the resulting magma.

**Lemma**

\( A(G, L, Q) \) is a loop of order \(|G|(|L| - 1) + 1\).

It is a Jordan loop if and only if

1. \((L, \ast)\) is a Jordan loop,
2. \((G, \cdot)\) and \((Q, \circ)\) are commutative, and
3. for every \(s, t \in Q\) either \(s \ast s = 1\) or

\[(s \ast s) \circ (t \circ s) = ((s \ast s) \circ t) \circ s\]
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Achievable orders using the Amalgam

Lemma

There exist NJLs of order $n \geq 6$ for each $n \neq 2^k + 1$.

Proof.

- $|G|$ must be odd. $|G| = 1$ is trivial. Rules out $n = 2^k + 1$.
- Otherwise, let $(L, \ast)$ and $(Q, \bullet)$ be groups.
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Orders of the form $n = 2^k + 1$

Using the finite model builder Mace4:

- there does not exist a NJL of order 9,
- searched unsuccessfully for a week for order 17, and
- quickly finds examples for orders less and greater than 17.
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Michael Kinyon, Kyle Pula, & Petr Vojtěchovský

Admissible Orders of Jordan Loops
Step 1  Construct a Jordan quasigroup of order $2^k$ that is covered by sub-quasigroups of order $2^{k-2}$.

Step 2  As before, extend to a loop of order $2^k + 1$ by replacing these sub-quasigroups with copies of a Jordan loop of order $2^{k-2} + 1$.

Step 3  Prove that the resulting loop is non-associative.

Fails to satisfy the Lagrange property except when $k < 3$ (i.e. $n = 5$ or $9$).
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### Theorem

There exist NJLs of order $n$ for all $n \geq 6$ and $n \neq 9$.

### Proof.

For $n \geq 6$ and $n \neq 9$, by construction.

For $n = 9$, consider two cases:

- $L$ is cyclic. Show that it is power associative and thus $\mathbb{Z}_9$.
- $L$ has exponent 3. Play Sudoku until you have $\mathbb{Z}_3 \times \mathbb{Z}_3$. 

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Main Result

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In any Jordan Loop,
- $x^3, x^4, \text{ and } x^5$ are well defined.
- $x^6$ need not be well defined.
- If $x^6$ is well defined, then $x^7$ and $x^8$ are well defined but $x^9$ need not be.
- $x^2x^k = x^{2+k}$ and $x^4x^k = x^{4+k}$

Construction

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