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Topological Ramsey spaces from Fraïssé classes, Ramsey-classification theorems, and initial structures in the Tukey types of p-points

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Abstract A general method for constructing a new class of topological Ramsey spaces is presented. Members of such spaces are infinite sequences of products of Fraïssé classes of finite relational structures satisfying the Ramsey property. The Product Ramsey Theorem of Sokič is extended to equivalence relations for finite products of structures from Fraïssé classes of finite relational structures satisfying the Ramsey property and the Order-Prescribed Free Amalgamation Property. This is essential to proving Ramsey-classification theorems for equivalence relations on fronts, generalizing the Pudlák–Rödl Theorem to this class of topological Ramsey spaces. To each topological Ramsey space in this framework corresponds an associated ultrafilter satisfying some weak partition property. By using the correct Fraïssé classes, we construct topological Ramsey spaces which are dense in the partial orders of Baum-

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gartner and Taylor (Trans Am Math Soc 241:283–309, 1978) generating p-points which are k-arrow but not $k+1$-arrow, and in a partial order of Blass (Trans Am Math Soc 179:145–166, 1973) producing a diamond shape in the Rudin-Keisler structure of p-points. Any space in our framework in which blocks are products of n many structures produces ultrafilters with initial Tukey structure exactly the Boolean algebra $\mathcal{P}(n)$. If the number of Fraïssé classes on each block grows without bound, then the Tukey types of the p-points below the space's associated ultrafilter have the structure exactly $[\omega]^{<\omega}$. In contrast, the set of isomorphism types of any product of finitely many Fraïssé classes of finite relational structures satisfying the Ramsey property and the OPFAP, partially ordered by embedding, is realized as the initial Rudin-Keisler structure of some p-point generated by a space constructed from our template.

Keywords Ultrafilter · Tukey · Rudin-Keisler · Ramsey theory · Topological Ramsey space

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1 Introduction

The Tukey theory of ultrafilters has recently seen much progress, developing into a full-fledged area of research drawing on set theory, topology, and Ramsey theory. Interest in Tukey reducibility on ultrafilters stems both from the fact that it is a weakening of the well-known Rudin-Keisler reducibility as well as the fact that it is a useful tool for classifying partial orderings.

Given ultrafilters $\mathcal{U}$, $\mathcal{V}$, we say that $\mathcal{U}$ is *Tukey reducible to* $\mathcal{V}$ (written $\mathcal{U} \leq_T \mathcal{V}$) if there is a function $f : \mathcal{V} \to \mathcal{U}$ which sends filter bases of $\mathcal{V}$ to filter bases of $\mathcal{U}$. We say that $\mathcal{U}$ and $\mathcal{V}$ are *Tukey equivalent* if both $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$. The collection of all ultrafilters Tukey equivalent to $\mathcal{U}$ is called the *Tukey type* of $\mathcal{U}$.

The question of which structures embed into the Tukey types of ultrafilters on the natural numbers was addressed to some extent in [10]. In that paper, the following were shown to be consistent with ZFC: chains of length $\mathfrak{c}$ embed into the Tukey types of p-points; diamond configurations embed into the Tukey types of p-points; and there are $2^{\mathfrak{c}}$ many Tukey-incomparable selective ultrafilters. However, [10] left open the question of which structures appear as *initial Tukey structures* in the Tukey types of ultrafilters, where by an initial Tukey structure we mean a collection of Tukey types of nonprincipal ultrafilters which is closed under Tukey reducibility.

The first progress in this direction was made in [24], where applying a canonical Ramsey theorem of Pudlák and Rödl (see Theorem 13), Todorcevic showed that every nonprincipal ultrafilter Tukey reducible to a Ramsey ultrafilter is in fact Tukey equivalent to that Ramsey ultrafilter. Thus, the initial Tukey structure below a Ramsey ultrafilter is simply a singleton.

Further progress on initial Tukey structures was made by Dobrinen and Todorcevic [11] and [12]. To each topological Ramsey space, there is a naturally associated ultrafilter obtained by forcing with the topological Ramsey space partially ordered modulo finite initial segments. The properties of the associated ultrafilters are inherited from the properties of the topological Ramsey space (see Sect. 4). In [11], a dense subset of a partial ordering of Laflamme from [17] which forces a weakly Ramsey ultrafilter was pared down to reveal the inner structure responsible for the desired properties to be that of a topological Ramsey space, $\mathcal{R}_1$. In fact, Laflamme's partial ordering is exactly that of an earlier example of Baumgartner and Taylor [2] (see Example 21). By proving and applying a new Ramsey classification theorem, generalizing the Pudlák–Rödl Theorem for canonical equivalence relations on barriers, it was shown in [11] that the ultrafilter associated with $\mathcal{R}_1$ has exactly one Tukey type of nonprincipal ultrafilters strictly below it, namely that of the projected Ramsey ultrafilter, and similarly for Rudin-Keisler reduction. Thus, the initial Tukey and Rudin-Keisler structures of nonprincipal ultrafilters reducible to the ultrafilter associated with $\mathcal{R}_1$ are both exactly a chain of length 2.

In [12], this work was extended to a new class of topological Ramsey spaces $\mathcal{R}_\alpha$, which are obtained as particular dense sets of forcings of Laflamme in [17]. In [12], it was proved that the structure of the Tukey types of ultrafilters Tukey reducible to the ultrafilter associated with $\mathcal{R}_\alpha$ is exactly a decreasing chain of order-type $\alpha + 1$. Likewise for the initial Rudin-Keisler structure. As before, this result was obtained by proving new Ramsey-classification theorems for canonical equivalence relations on barriers and applying them to deduce the Tukey and Rudin-Keisler structures below the ultrafilter associated with $\mathcal{R}_\alpha$.

All of the results in [11,24] and [12] produced initial Tukey and Rudin-Keisler structures which are linear orders, precisely, decreasing chains of some countable successor ordinal length. This led to the following questions, which motivated the present and forthcoming work.

Question 1 What are the possible initial Tukey structures for ultrafilters on a countable base set?

Question 2 What are the possible initial Rudin-Keisler structures for ultrafilters on a countable base set?

Question 3 For a given ultrafilter $\mathcal{U}$, what is the structure of the Rudin-Keisler ordering of the isomorphism classes of ultrafilters Tukey reducible to $\mathcal{U}$?

Question 3 was answered in [11,12,24] by showing that each Tukey type below the associated ultrafilter consists of iterated Fubini products of p-points obtained from projections of the ultrafilter forced by the space.

Related to these questions are the following two motivating questions. Before [12], there were relatively few examples in the literature of topological Ramsey spaces. The constructions in that paper led to considering what other new topological Ramsey spaces can be formed. Our general construction method presented in Sect. 3 is a step toward answering the following larger question.

Question 4 What general construction schemes are there for constructing new topological Ramsey spaces?

We point out some recent work in this vein constructing new types of topological Ramsey spaces. Mijares and Padilla [19] construct new spaces of infinite polyhedra, and Mijares and Torrealba [20] construct spaces whose members are countable metric spaces with rational valued metrics. These spaces answer questions in Ramsey theory regarding homogeneous structures and random objects. One of aims of the present work is to find a general framework for ultrafilters satisfying partition properties in terms of topological Ramsey spaces. See also [8,9] for new construction schemes.

Question 5 Is each ultrafilter on some countable base satisfying some partition relations actually an ultrafilter associated with some topological Ramsey space (or something close to a topological Ramsey space)? Is there some general framework of topological Ramsey spaces into which many or all examples of ultrafilters with partition properties fit?

Some recent work of Dobrinen [8] constructs high-dimensional extensions of the Ellentuck space. These topological Ramsey spaces generate ultrafilters which are not p-points but which have strong partition properties; precisely these spaces yield the ultrafilters generic for the forcings $\mathcal{P}(\omega^n)/\mathrm{Fin}^{\otimes n}$, $2 \leq n < \omega$. The structure of the spaces aids in finding their initial Tukey structures via new extensions of the Pudlák–Rödl Theorem for these spaces.

It turns out that whenever an ultrafilter is associated with some topological Ramsey space, the ultrafilter has complete combinatorics, meaning that in the presence of a supercompact cardinal, the ultrafilter is generic over $L(\mathbb{R})$. This was proved by Di Prisco et al. [4], building on work of Todorcevic [15] for the Ellentuck space. Thus, finding a general framework for ultrafilters with partition properties in terms of ultrafilters associated with topological Ramsey spaces has the benefit of providing a large class of forcings with complete combinatorics.

In this paper we provide a general scheme for constructing new topological Ramsey spaces. This construction scheme uses products of finite ordered relational structures from Fraïssé classes with the Ramsey property. The details are set out in Sect. 3. The goal of this construction scheme is several-fold. We aim to construct topological Ramsey spaces with associated ultrafilters which have initial Tukey structures which are not simply linear orders. This is achieved by allowing "blocks" of the members of the Ramsey space to consist of products of structures, rather than trees as was the case in [12]. In particular, for each $n < \omega$, we construct a hypercube space $\mathcal{H}^n$ which produces an ultrafilter with initial Tukey and Rudin-Keisler structures exactly that of the Boolean algebra $\mathcal{P}(n)$. See Example 24 and Theorems 60 and 67.

We also seek to use topological Ramsey spaces to provide a unifying framework for p-points satisfying weak partition properties. This is the focus in Sect. 4. All of the p-points of Baumgartner and Taylor [2] fit into our scheme, in particular, the k-arrow, not $(k + 1)$-arrow p-points which they construct. In the other direction, for many collections of weak partition properties, we show there is a topological Ramsey space with associated ultrafilter simultaneously satisfying those properties.

The general Ramsey-classification Theorem 38 in Sect. 6 hinges on Theorem 31 in Sect. 5, which generalizes the Erdős-Rado Theorem (see Theorem 11) in two ways: by extending it from finite linear orders to Fraïssé classes of finite ordered relational structures with the Ramsey property and the Order-Prescribed Free Amalgamation Property (see Definition 29), and by extending it to finite products of members of such classes. Theorem 31 also extends the Product Ramsey Theorem of Sokič (see Theorem

14) from finite colorings to equivalence relations, but at the expense of restricting to a certain subclass of those Fraïssé classes for which his theorem holds. Theorem 31 is applied in Sect. 6 to prove Theorem 38, which generalizes the Ramsey-classification theorems in [11] for equivalence relations on fronts to the setting of the topological Ramsey spaces in this paper. Furthermore, we show that the Abstract Nash-Williams Theorem (as opposed to the Abstract Ellentuck Theorem) suffices for the proof.

Section 7 contains theorems general to all topological Ramsey spaces $(\mathcal{R}, \leq, r)$, not just those constructed from a generating sequence. In this section, general notions of a filter being selective or Ramsey for the space $\mathcal{R}$ are put forth. The main result of this section, Theorem 56, shows that Tukey reductions for ultrafilters Ramsey for a topological Ramsey space can be assumed to be continuous with respect to the metric topology on the Ramsey space. In particular, it is shown that any cofinal map from an ultrafilter Ramsey for $\mathcal{R}$ is continuous on some base for that ultrafilter, and even better, is *basic* (see Definition 48). This section also contains a general method for analyzing ultrafilters Tukey reducible to some ultrafilter Ramsey for $\mathcal{R}$ in terms of fronts and canonical functions. (See Proposition 50 and neighboring text.)

Theorems 38 and 56 are applied in Sect. 8 to answer Questions 1 - 3. All initial Tukey and Rudin-Keisler structures associated with the ultrafilters generated by the class of topological Ramsey spaces constructed in this paper are found. Theorem 60, shows that whenever n Fraïssé classes are used to generate a topological Ramsey space, then the initial Tukey structure below the associated ultrafilter is exactly the Boolean algebra $\mathcal{P}(n)$. When infinitely many Fraïssé classes are used, then the initial Tukey structure of the p-points below the associated forced filter is exactly $([\omega]^{<\omega}, \subseteq)$. In Theorem 66, we find the exact structure of the Rudin-Keisler types inside the Tukey types of ultrafilters reducible to the associated filter. Theorem 67 shows that if $\mathcal{R}$ is a topological Ramsey space constructed from some Fraïssé classes $\mathcal{K}_j$, $j \in J$, and $\mathcal{C}$ is a Ramsey filter on $(\mathcal{R}, \leq)$, then the Rudin-Keisler ordering of the p-points Tukey reducible to $\mathcal{C}$ is isomorphic to the collection of all (equivalence classes of) finite products of members of the classes $\mathcal{K}_j$, partially ordered under embeddability.

Attributions The work in Sects. 3, 4 and 5 is due to Dobrinen. Section 6 comprises joint work of Dobrinen and Mijares. Sections 7 and 8 are joint work of Dobrinen and Trujillo, building on some of the work in Trujillo's thesis. The main results in this paper for the special case of the space $\mathcal{H}^2$ constitute work of Trujillo in his PhD thesis [27].

2 Background on topological Ramsey spaces, notation, and classical canonization theorems

Todorcevic [26] distills the key properties of the Ellentuck space into four axioms which guarantee that a space is a topological Ramsey space. For the sake of clarity, we reproduce his definitions here. The following can all be found at the beginning of Chapter 5 in [26].

The axioms **A.1** –**A.4** are defined for triples $(\mathcal{R}, \leq, r)$ of objects with the following properties. $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-ordering on $\mathcal{R}$, and $r : \mathcal{R} \times \omega \to \mathcal{AR}$ is a mapping giving us the sequence $(r_n(\cdot) = r(\cdot, n))$ of approximation mappings, where $\mathcal{AR}$ is the collection of all finite approximations to members of $\mathcal{R}$. For $a \in \mathcal{AR}$ and

$A, B \in \mathcal{R}$,

$$[a, B] = \{A \in \mathcal{R} : A \le B \text{ and } (\exists n)\, r_n(A) = a\}. \tag{1}$$

For $a \in \mathcal{AR}$, let $|a|$ denote the length of the sequence a. Thus, $|a|$ equals the integer k for which $a = r_k(a)$. For $a, b \in \mathcal{AR}$, $a \sqsubseteq b$ if and only if $a = r_m(b)$ for some $m \le |b|$. $a \sqsubset b$ if and only if $a = r_m(b)$ for some $m < |b|$. For each $n < \omega$, $\mathcal{AR}_n = \{r_n(A) : A \in \mathcal{R}\}$.

A.1 (a) $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$.

(b) $A \ne B$ implies $r_n(A) \ne r_n(B)$ for some n.

(c) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

A.2 There is a quasi-ordering $\le_{\text{fin}}$ on $\mathcal{AR}$ such that

(a) $\{a \in \mathcal{AR} : a \le_{\text{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,

(b) $A \le B$ iff $(\forall n)(\exists m)\, r_n(A) \le_{\text{fin}} r_m(B)$,

(c) $\forall a, b, c \in \mathcal{AR}[a \sqsubset b \wedge b \le_{\text{fin}} c \to \exists d \sqsubset c\; a \le_{\text{fin}} d]$.

We abuse notation and for $a \in \mathcal{AR}$ and $A \in \mathcal{R}$, we write $a \le_{\text{fin}} A$ to denote that there is some $n < \omega$ such that $a \le_{\text{fin}} r_n(A)$. $\text{depth}_B(a)$ denotes the least n, if it exists, such that $a \le_{\text{fin}} r_n(B)$. If such an n does not exist, then we write $\text{depth}_B(a) = \infty$. If $\text{depth}_B(a) = n < \infty$, then $[\text{depth}_B(a), B]$ denotes $[r_n(B), B]$.

A.3 (a) If $\text{depth}_B(a) < \infty$ then $[a, A] \ne \emptyset$ for all $A \in [\text{depth}_B(a), B]$.

(b) $A \le B$ and $[a, A] \ne \emptyset$ imply that there is $A' \in [\text{depth}_B(a), B]$ such that $\emptyset \ne [a, A'] \subseteq [a, A]$.

If $n > |a|$, then $r_n[a, A]$ denotes the collection of all $b \in \mathcal{AR}_n$ such that $a \sqsubset b$ and $b \le_{\text{fin}} A$.

A.4 If $\text{depth}_B(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$, then there is $A \in [\text{depth}_B(a), B]$ such that $r_{|a|+1}[a, A] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, A] \subseteq \mathcal{O}^c$.

The topology on $\mathcal{R}$ is given by the basic open sets $[a, B]$. This topology is called the *Ellentuck* topology on $\mathcal{R}$; it extends the usual metrizable topology on $\mathcal{R}$ when we consider $\mathcal{R}$ as a subspace of the Tychonoff cube $\mathcal{AR}^{\mathbb{N}}$. Given the Ellentuck topology on $\mathcal{R}$, the notions of nowhere dense, and hence of meager are defined in the natural way. Thus, we may say that a subset $\mathcal{X}$ of $\mathcal{R}$ has the *property of Baire* iff $\mathcal{X} = \mathcal{O} \cap \mathcal{M}$ for some Ellentuck open set $\mathcal{O} \subseteq \mathcal{R}$ and Ellentuck meager set $\mathcal{M} \subseteq \mathcal{R}$.

Definition 6 ([26]) *A subset $\mathcal{X}$ of $\mathcal{R}$ is* Ramsey *if for every $\emptyset \ne [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}$ is* Ramsey null *if for every $\emptyset \ne [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.*

A triple $(\mathcal{R}, \le, r)$ is a topological Ramsey space *if every property of Baire subset of $\mathcal{R}$ is Ramsey and if every meager subset of $\mathcal{R}$ is Ramsey null.*

The following result can be found as Theorem 5.4 in [26].

Theorem 7 (Abstract Ellentuck Theorem) *If $(\mathcal{R}, \le, r)$ is closed (as a subspace of $\mathcal{AR}^{\mathbb{N}}$) and satisfies axioms* **A.1***,* **A.2***,* **A.3***, and* **A.4***, then every property of Baire* *subset of $\mathcal{R}$ is Ramsey, and every meager subset is Ramsey null; in other words, the triple $(\mathcal{R}, \leq, r)$ forms a topological Ramsey space.*

For a topological Ramsey space, certain types of subsets of the collection of approximations $\mathcal{AR}$ have the Ramsey property.

Definition 8 ([26]) *A family $\mathcal{F} \subseteq \mathcal{AR}$ of finite approximations is*

(1) Nash-Williams *if $a \not\sqsubseteq b$ for all $a \neq b \in \mathcal{F}$;*
(2) Sperner *if $a \not\leq_{\text{fin}} b$ for all $a \neq b \in \mathcal{F}$;*
(3) Ramsey *if for every partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ and every $X \in \mathcal{R}$, there are $Y \leq X$ and $i \in \{0, 1\}$ such that $\mathcal{F}_i | Y = \emptyset$.*

The Abstract Nash-Williams Theorem (Theorem 5.17 in [26]), which follows from the Abstract Ellentuck Theorem, will suffice for the arguments in this paper.

Theorem 9 (Abstract Nash-Williams Theorem) *Suppose $(\mathcal{R}, \leq, r)$ is a closed triple that satisfies* **A.1**–**A.4**. *Then every Nash-Williams family of finite approximations is Ramsey.*

Definition 10 *Suppose $(\mathcal{R}, \leq, r)$ is a closed triple that satisfies* **A.1**–**A.4**. *Let $X \in \mathcal{R}$. A family $\mathcal{F} \subseteq \mathcal{AR}$ is a* front *on $[0, X]$ if*

(1) For each $Y \in [0, X]$, there is an $a \in \mathcal{F}$ such that $a \sqsubset Y$; and
(2) $\mathcal{F}$ is Nash-Williams.

$\mathcal{F}$ is a barrier *if (1) and (2′) hold, where*

(2′) $\mathcal{F}$ is Sperner.

The quintessential example of a topological Ramsey space is the *Ellentuck space*, which is the triple $([\omega]^\omega, \subseteq, r)$. Members $X \in [\omega]^\omega$ are considered as infinite increasing sequences of natural numbers, $X = \{x_0, x_1, x_2, \dots\}$. For each $n < \omega$, the n-th approximation to X is $r_n(X) = \{x_i : i < n\}$; in particular, $r_0(X) = \emptyset$. The basic open sets of the Ellentuck topology are sets of the form $[a, X] = \{Y \in [\omega]^\omega : a \sqsubset Y$ and $Y \subseteq X\}$. Notice that the Ellentuck topology is finer than the metric topology on $[\omega]^\omega$.

In the case of the Ellentuck space, the Abstract Ellentuck Theorem says the following: Whenever a subset $\mathcal{X} \subseteq [\omega]^\omega$ has the property of Baire in the Ellentuck topology, then that set is *Ramsey*, meaning that every open set contains a basic open set either contained in $\mathcal{X}$ or else disjoint from $\mathcal{X}$. This was proved by Ellentuck [13].

The first theorem to extend Ramsey's Theorem from finite-valued functions to countably infinite-valued functions was a theorem of Erdős and Rado. They found that in fact, given any equivalence relation on $[\omega]^n$, there is an infinite subset on which the equivalence relation is canonical—one of exactly 2^n many equivalence relations. We shall state the finite version of their theorem, as it is all that is used in this paper (see Sect. 5).

Let $n \leq l$. For each $I \subseteq n$, the equivalence relation E_I on $[l]^n$ is defined as follows: For $b, c \in [l]^n$,

$$b \; E_I \; c \iff \forall i \in I (b_i = c_i),$$

where $\{b_0, \ldots, b_{n-1}\}$ and $\{c_0, \ldots, c_{n-1}\}$ are the strictly increasing enumerations of b and c, respectively. An equivalence relation E on $[l]^n$ is *canonical* if and only if there is some $I \subseteq n$ for which $E = E_I$.

Theorem 11 (Finite Erdős-Rado Theorem, [14]) *Given $n \leq l$, there is an $m > l$ such that for each equivalence relation E on $[m]^n$, there is a subset $s \subseteq m$ of cardinality l such that $E \restriction [s]^n$ is canonical; that is, there is a set $I \subseteq n$ such that $E \restriction [s]^n = E_I \restriction [s]^n$.*

Pudlák and Rödl later extended this theorem to equivalence relations on general barriers on the Ellentuck space. To state their theorem, we need the following definition.

Definition 12 *A map φ from a front $\mathcal{F}$ on the Ellentuck space into ω is called* irreducible *if*

(1) φ *is* inner, *meaning that $\varphi(a) \subseteq a$ for all $a \in \mathcal{F}$; and*
(2) φ *is* Nash-Williams, *meaning that $\varphi(a) \not\sqsubset \varphi(b)$ for all $a, b \in \mathcal{F}$ such that $\varphi(a) \neq \varphi(b)$.*

Given a front $\mathcal{F}$ and an $X \in [\omega]^\omega$, we let $\mathcal{F} \restriction X$ denote $\{a \in \mathcal{F} : a \subseteq X\}$. Given an equivalence relation E on a barrier $\mathcal{F}$, we say that an irreducible map φ *represents* E on $\mathcal{F} \restriction X$ if for all $a, b \in \mathcal{F} \restriction X$, we have $a \; E \; b \leftrightarrow \varphi(a) = \varphi(b)$.

The following theorem of Pudlák and Rödl is the basis for all subsequent canonization theorems for fronts on the general topological Ramsey spaces considered in the papers [11,12].

Theorem 13 *(Pudlák/Rödl, [23]) For any barrier $\mathcal{F}$ on the Ellentuck space and any equivalence relation on $\mathcal{F}$, there is an $X \in [\omega]^\omega$ and an irreducible map φ such that the equivalence relation restricted to $\mathcal{F} \restriction X$ is represented by φ.*

Theorem 13 was generalized to a class of topological Ramsey spaces whose members are trees in [11,12]. In Sect. 6, we shall generalize this theorem to the broad class of topological Ramsey spaces defined in the next section.

3 A general method for constructing topological Ramsey spaces using Fraïssé theory

We review only the facts of Fraïssé theory for ordered relational structures which are necessary to this article. More general background on Fraïssé theory can be found in [16]. We shall call $L = \{<\} \cup \{R_i\}_{i \in I}$ an *ordered relational signature* if it consists of the order relation symbol $<$ and a (countable) collection of *relation symbols* R_i, where for each $i \in I$, we let $n(i)$ denote the *arity* of R_i. A *structure* for L is of the form $\boldsymbol{A} = \langle |\boldsymbol{A}|, <^{\boldsymbol{A}}, \{R_i^{\boldsymbol{A}}\}_{i \in I} \rangle$, where $|\boldsymbol{A}| \neq \emptyset$ is the *universe* of $\boldsymbol{A}$, $<^{\boldsymbol{A}}$ is a linear ordering of $|\boldsymbol{A}|$, and for each $i \in I$, $R_i^{\boldsymbol{A}} \subseteq A^{n(i)}$. An *embedding* between structures $\boldsymbol{A}, \boldsymbol{B}$ for L is an injection $\iota : |\boldsymbol{A}| \rightarrow |\boldsymbol{B}|$ such that for any two $a, a' \in |\boldsymbol{A}|$, $a <^{\boldsymbol{A}} a' \leftrightarrow \iota(a) <^{\boldsymbol{B}} \iota(a')$, and for all $i \in I$, $R_i^{|\boldsymbol{A}|}(a_1, \ldots, a_{n(i)}) \leftrightarrow R_i^{|\boldsymbol{B}|}(\iota(a_1), \ldots, \iota(a_{n(i)}))$. If ι is the identity map, then we say that $\boldsymbol{A}$ is a *substructure* of $\boldsymbol{B}$. We say that ι is an *isomorphism* if ι is an onto embedding. We write $\boldsymbol{A} \leq \boldsymbol{B}$ to denote that $\boldsymbol{A}$ can be embedded into $\boldsymbol{B}$; and we write $\boldsymbol{A} \cong \boldsymbol{B}$ to denote that $\boldsymbol{A}$ and $\boldsymbol{B}$ are isomorphic.

A class $\mathcal{K}$ of finite structures for an ordered relational signature L is called *hereditary* if whenever $\boldsymbol{B} \in \mathcal{K}$ and $\boldsymbol{A} \leq \boldsymbol{B}$, then also $\boldsymbol{A} \in \mathcal{K}$. $\mathcal{K}$ satisfies the *joint embedding property* if for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, there is a $\boldsymbol{C} \in \mathcal{K}$ such that $\boldsymbol{A} \leq \boldsymbol{C}$ and $\boldsymbol{B} \leq \boldsymbol{C}$. We say that $\mathcal{K}$ satisfies the *amalgamation property* if for any embeddings $f : \boldsymbol{A} \to \boldsymbol{B}$ and $g : \boldsymbol{A} \to \boldsymbol{C}$, with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$, there is a $\boldsymbol{D} \in \mathcal{K}$ and there are embeddings $r : \boldsymbol{B} \to \boldsymbol{D}$ and $s : \boldsymbol{C} \to \boldsymbol{D}$ such that $r \circ f = s \circ g$. $\mathcal{K}$ satisfies the *strong amalgamation property* there are embeddings $r : \boldsymbol{B} \to \boldsymbol{D}$ and $s : \boldsymbol{C} \to \boldsymbol{D}$ such that $r \circ f = s \circ g$ and additionally, $r(\boldsymbol{B}) \cap s(\boldsymbol{C}) = r \circ f(\boldsymbol{A}) = s \circ g(\boldsymbol{A})$. A class of finite structures $\mathcal{K}$ is called a *Fraïssé class of ordered relational structures* for an ordered relational signature L if it is hereditary, satisfies the joint embedding and amalgamation properties, contains (up to isomorphism) only countably many structures, and contains structures of arbitrarily large finite cardinality.

Let $\mathcal{K}$ be a hereditary class of finite structures for an ordered relational signature L. For $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ with $\boldsymbol{A} \leq \boldsymbol{B}$, we use $\binom{\boldsymbol{B}}{\boldsymbol{A}}$ to denote the set of all substructures of $\boldsymbol{B}$ which are isomorphic to $\boldsymbol{A}$. Given structures $\boldsymbol{A} \leq \boldsymbol{B} \leq \boldsymbol{C}$ in $\mathcal{K}$, we write

$$\boldsymbol{C} \to (\boldsymbol{B})_k^{\boldsymbol{A}}$$

to denote that for each coloring of $\binom{\boldsymbol{C}}{\boldsymbol{A}}$ into k colors, there is a $\boldsymbol{B}' \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ such that $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$ is homogeneous, i.e. monochromatic, meaning that every member of $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$ has the same color. We say that $\mathcal{K}$ has the *Ramsey property* if and only if for any two structures $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$ and any natural number $k \geq 2$, there is a $\boldsymbol{C} \in \mathcal{K}$ with $\boldsymbol{B} \leq \boldsymbol{C}$ such that $\boldsymbol{C} \to (\boldsymbol{B})_k^{\boldsymbol{A}}$.

For finitely many Fraïssé classes $\mathcal{K}_j$, $j \in J$ for some $J < \omega$, we write $\binom{(\boldsymbol{B}_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$ to denote the set of all sequences $(\boldsymbol{D}_j)_{j\in J}$ such that for each $j \in J$, $\boldsymbol{D}_j \in \binom{\boldsymbol{B}_j}{\boldsymbol{A}_j}$. For structures $\boldsymbol{A}_j \leq \boldsymbol{B}_j \leq \boldsymbol{C}_j \in \mathcal{K}_j$, $j \in J$, we write

$$(\boldsymbol{C}_j)_{j\in J} \to ((\boldsymbol{B}_j)_{j\in J})_k^{(\boldsymbol{A}_j)_{j\in J}}$$

to denote that for each coloring of the members of $\binom{(\boldsymbol{C}_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$ into k colors, there is $(\boldsymbol{B}'_j)_{j\in J} \in \binom{(\boldsymbol{C}_j)_{j\in J}}{(\boldsymbol{B}_j)_{j\in J}}$ such that all members of $\binom{(\boldsymbol{B}'_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$ have the same color; that is, the set $\binom{(\boldsymbol{B}'_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$ is homogeneous. We subscribe to the usual convention that when no k appears in the expression, it is assumed that $k = 2$.

We point out that by Theorem A of Nešetřil and Rödl [21], there is a large class of Fraïssé classes of finite ordered relational structures with the Ramsey property. In particular, the collection of all finite linear orderings, the collection of all finite ordered n-clique free graphs, and the collection of all finite ordered complete graphs are examples of Fraïssé classes fulfilling our requirements. Moreover, finite products of members of such classes preserve the Ramsey property, as we now see. The following theorem for products of Ramsey classes of finite objects is due to Sokić and can be found in his PhD thesis.

Theorem 14 *(Product Ramsey Theorem, Sokić [25]) Let s and k be fixed natural numbers and let $\mathcal{K}_j$, $j \in s$, be a sequence of Ramsey classes of finite objects. Fix two sequences $(\boldsymbol{B}_j)_{j\in s}$ and $(\boldsymbol{A}_j)_{j\in s}$ such that for each $j \in s$, we have $\boldsymbol{A}_j, \boldsymbol{B}_j \in \mathcal{K}_j$ and $\boldsymbol{A}_j \leq \boldsymbol{B}_j$. Then there is a sequence $(\boldsymbol{C}_j)_{j\in s}$ such that $\boldsymbol{C}_j \in \mathcal{K}_j$ for each $j \in s$, and*

$$(\boldsymbol{C}_j)_{j\in s} \to ((\boldsymbol{B})_{j\in s})_k^{(\boldsymbol{A}_j)_{j\in s}}.$$

We now present our notion of a *generating sequence*. Such sequences will be used to generate new topological Ramsey spaces.

Definition 15 (Generating sequence) *Let $1 \leq J \leq \omega$ and $\mathcal{K}_j$, $j \in J$, be a collection of Fraïssé classes of finite ordered relational structures with the Ramsey property. For each $k \in \omega$, if $J < \omega$ then let $J_k = J$, and if $J = \omega$ then let $J_k = k + 1$.*

For each $k < \omega$ and $j \in J_k$, suppose $\mathbf{A}_{k,j}$ is some fixed member of $\mathcal{K}_j$, and let $\mathbf{A}_k$ denote the sequence $(\mathbf{A}_{k,j})_{j\in J_k}$. We say that $\langle \mathbf{A}_k : k < \omega\rangle$ is a generating sequence *if and only if*

(1) For each $j \in J_0$, $|\mathbf{A}_{0,j}| = 1$.
(2) For each $k < \omega$ and all $j \in J_k$, $\mathbf{A}_{k,j}$ is a substructure of $\mathbf{A}_{k+1,j}$.
(3) For each $j \in J$ and each structure $\boldsymbol{B} \in \mathcal{K}_j$, there is a k such that $\boldsymbol{B} \leq \mathbf{A}_{k,j}$.
(4) (Pigeonhole) For each pair $k < m < \omega$, there is an $n > m$ such that

$$(\mathbf{A}_{n,j})_{j\in J_k} \to {(\mathbf{A}_{m,j})_{j\in J_k}}^{(\mathbf{A}_{k,j})_{j\in J_k}}.$$

Remark Note that (3) implies that for each $j \in J$ and each $\boldsymbol{B} \in \mathcal{K}_j$, $\boldsymbol{B} \leq \mathbf{A}_{k,j}$ for all but finitely many k.

We now define the new class of topological Ramsey spaces which are the focus of this article.

Definition 16 (The spaces $\mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)$) *Let $1 \leq J \leq \omega$ and $\mathcal{K}_j$, $j \in J$, be a collection of Fraïssé classes of finite ordered relational structures with the Ramsey property. Let $\langle \mathbf{A}_k : k < \omega\rangle$ be any generating sequence. Let $\mathbb{A} = \langle\langle k, \mathbf{A}_k\rangle : k < \omega\rangle$. $\mathbb{A}$ is the maximal member of $\mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)$.*

We define B to be a member of $\mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)$ if and only if $B = \langle\langle n_k, \boldsymbol{B}_k\rangle : k < \omega\rangle$, where

(1) $(n_k)_{k<\omega}$ is some strictly increasing sequence of natural numbers; and
(2) For each $k < \omega$, $\boldsymbol{B}_k$ is some sequence $(\boldsymbol{B}_{k,j})_{j\in J_k}$, where for each $j \in J_k$, $\boldsymbol{B}_{k,j} \in \binom{\mathbf{A}_{n_k,j}}{\mathbf{A}_{k,j}}$.

We use $B(k)$ to denote $\langle n_k, \boldsymbol{B}_k\rangle$, the k-th block *of B. Let $\mathcal{R}(k)$ denote $\{B(k) : B \in \mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)\}$, the collection of all k-th blocks of members of $\mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)$. The n-th approximation of B is $r_n(B) := \langle B(0), \ldots, B(n-1)\rangle$. In particular, $r_0(B) = \emptyset$. Let $\mathcal{AR}_n = \{r_n(B) : B \in \mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)\}$, the collection of all n-th approximations to members of $\mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)$. Let $\mathcal{AR} = \bigcup_{n<\omega} \mathcal{AR}_n$, the collection of all finite approximations to members of $\mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)$.*

Define the partial order $\le$ *on* $\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega\rangle)$ *as follows. For* $B = \langle\langle m_k, \boldsymbol{B}_k\rangle : k < \omega\rangle$ *and* $C = \langle\langle n_k, \boldsymbol{C}_k\rangle : k < \omega\rangle$, *define* $C \le B$ *if and only if for each* k *there is an* l_k *such that* $n_k = m_{l_k}$ *and for all* $j \in J_k$, $\boldsymbol{C}_{k,j} \in \binom{\boldsymbol{B}_{l_k,j}}{\boldsymbol{A}_{k,j}}$.

Define the partial order $\le_{\text{fin}}$ *on* $\mathcal{AR}$ *as follows: For* $b = \langle\langle m_k, \boldsymbol{B}_k\rangle : k < p\rangle$ *and* $c = \langle\langle n_k, \boldsymbol{C}_k\rangle : k < q\rangle$, *define* $c \le_{\text{fin}} b$ *if and only if there are* $C \le B$ *and* $k \le l$ *such that* $c = r_q(C)$, $b = r_p(B)$, *and for each* $k < q$, $n_k = m_{l_k}$ *for some* $l_k < p$.

For $c \in \mathcal{AR}$ and $B \in \mathcal{R}$, $\text{depth}_B(c)$ denotes the minimal d such that $c \le_{\text{fin}} r_d(B)$, if such a d exists; otherwise $\text{depth}_B(c) = \infty$. Note that for $c = \langle\langle n_k, \boldsymbol{C}_k\rangle : k < q\rangle$, $\text{depth}_{\mathbb{A}}(c)$ is equal to $n_{q-1} + 1$. The *length* of c, denoted by $|c|$, is the minimal q such that $c = r_q(c)$. For $b, c \in \mathcal{AR}$, we write $b \sqsubseteq c$ if and only if there is a $p \le |c|$ such that $b = r_p(c)$. In this case, we say that b is an *initial segment* of c. We use $b \sqsubset c$ to denote that b is a proper initial segment of c; that is $b \sqsubseteq c$ and $b \ne c$.

Remark The members of $\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega\rangle)$ are infinite squences B which are isomorphic to the maximal member $\mathbb{A}$, in the sense that for each k-th block $B(k) = \langle n_k, \boldsymbol{B}_k\rangle$, each of the structures $\boldsymbol{B}_{k,j}$ is isomorphic to $\boldsymbol{A}_{k,j}$. This idea, of forming a topological Ramsey space by taking the collection of all infinite sequences coming from within some fixed sequence and preserving the same form as this fixed sequence, is extracted from the Ellentuck space itself, and was first extended to more generality in [11].

The above method of construction yields a new class of topological Ramsey spaces. The proof below is jointly written with Trujillo.

Theorem 17 *Let* $1 \le J \le \omega$ *and* $\mathcal{K}_j$, $j \in J$, *be a collection of Fraïssé classes of finite ordered relational structures with the Ramsey property. For each generating sequence* $\langle \boldsymbol{A}_k : k < \omega\rangle$, *the space* $(\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega\rangle), \le, r)$ *satisfies axioms* **A.1** –**A.4** and is closed in $\mathcal{AR}^\omega$, and hence, is a topological Ramsey space.

Proof Let $\mathcal{R}$ denote $\mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)$. $\mathcal{R}$ is identified with the subspace of the Tychonov power $\mathcal{AR}^\omega$ consisting of all sequences $\langle a_n, n < \omega\rangle$ for which there is a $B \in \mathcal{R}$ such that for each $n < \omega$, $a_n = r_n(B)$. $\mathcal{R}$ forms a closed subspace of $\mathcal{AR}^\omega$, since for each sequence $\langle a_n, n < \omega\rangle$ with the properties that each $a_n \in \mathcal{AR}_n$ and $a_n \sqsubset a_{n+1}$, then $\big\langle\langle \text{depth}_A(a_{n+1}), a_{n+1}(n)\rangle : n < \omega\big\rangle$ is a member of $\mathcal{R}$. It is routine to check that axioms **A.1** and **A.2** hold.

A.3 (1) If $\text{depth}_B(a) = n < \omega$, then $a \le_{\text{fin}} r_n(B)$. If $C \in [\text{depth}_B(a), B]$, then $r_n(B) = r_n(C)$ and for each $k > n$, there is an m_k such that $(\mathbf{C}_{m_k,j})_{j\in J_k} \le (\mathbf{B}_{k,j})_{j\in J_k}$. For each $i \ge |a|$, let $D(i)$ be an element of $\mathcal{R}(i)$ such that $(\mathbf{D}_{i,j})_{j\in J_i}$ is a substructure of $(\mathbf{C}_{i,j})_{j\in J_i}$ isomorphic to $(\mathbf{A}_{i,j})_{j\in J_i}$. Let $D = a^\frown \langle D(i) : |a| \le i < \omega\rangle \in \mathcal{R}$. Then $D \in [a, B]$, so $[a, B] \ne \emptyset$.

(2) Suppose that $B \le C$ and $[a, B] \ne \emptyset$. Let $n = \text{depth}_C(a)$. Then $n < \infty$ since $B \le C$. Let $D = r_n(C)^\frown \langle B(n+i) : i < \omega\rangle$. Then $D \in [\text{depth}_C(a), C]$ and $\emptyset \ne [a, D] \subseteq [a, B]$.

A.4 Suppose that $B = \langle(n_k, \mathbf{B}_k) : k < \omega\rangle$, $\text{depth}_B(a) < \infty$, and $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$. Let $n = |a|$. By (4) in the definition of a generating sequence, there is a strictly increasing sequence $(k_i)_{i\ge n}$ such that $(\boldsymbol{A}_{k_i,j})_{j\in J_n} \to \binom{(\boldsymbol{A}_{i,j})_{j\in J_n}}{(\boldsymbol{A}_{n,j})_{j\in J_n}}$, for each $i \ge n$. For each $i \ge n$, choose some $(\boldsymbol{C}_{i,j})_{j\in J_i}$ in $\binom{(\boldsymbol{B}_{k_i,j})_{j\in J_i}}{(\boldsymbol{A}_{i,j})_{j\in J_i}}$ such that the collection

$$\left\{ \langle n_{k_i}, (\boldsymbol{X}_{i,j})_{j\in J_n}\rangle : (\boldsymbol{X}_{i,j})_{j\in J_n} \in \binom{(\boldsymbol{C}_{i,j})_{j\in J_n}}{(\boldsymbol{A}_{n,j})_{j\in J_n}} \right\}$$

is homogeneous for $\mathcal{O}$. Infinitely many of these $(\boldsymbol{C}_{i,j})_{j\in J_i}$ will agree about being in or out of $\mathcal{O}$. Thus, for some subsequence $(k_{i_l})_{l\geq n}$, there are $(\boldsymbol{D}_{l,j})_{j\in J_l} \in \binom{(\boldsymbol{C}_{i_l,j})_{j\in J_l}}{(\boldsymbol{A}_{l,j})_{j\in J_l}}$ such that letting $D = a^\frown\langle\langle n_{k_{i_l}}, (\boldsymbol{D}_{l,j})_{j\in J_l}\rangle : l \geq n\rangle$, we have that $r_{n+1}[a, D]$ is either contained in or disjoint from $\mathcal{O}$. □

We fix the following notation, which is used throughout this paper.

Notation 1 For $a \in \mathcal{AR}$ and $B \in \mathcal{R}$, we write $a \leq_{\text{fin}} B$ to mean that there is some $A \in \mathcal{R}$ such that $A \leq B$ and $a = r_n(A)$ for some n. For $\mathcal{H} \subseteq \mathcal{AR}$ and $B \in \mathcal{R}$, let $\mathcal{H}|B$ denote the collection of all $a \in \mathcal{H}$ such that $a \leq_{\text{fin}} B$.

For $n < \omega$, $\mathcal{R}(n) = \{C(n) : C \in \mathcal{R}\}$, and $\mathcal{R}(n)|B = \{C(n) : C \leq B\}$. B/a denotes the tail of B which is above every block in a. $\mathcal{R}(n)|B/a$ denotes the members of $\mathcal{R}(n)|B$ which are above a.

4 Ultrafilters associated with topological Ramsey spaces constructed from generating sequences and their partition properties

In this section, we show that many examples of ultrafilters satisfying partition properties can be seen to arise as ultrafilters associated with some topological Ramsey spaces constructed from a generating sequence. In particular, the ultrafilters of Baumgartner and Taylor in Section 4 of [2] arising from norms fit into this framework. We begin by reviewing some important types of ultrafilters. All of the following definitions can found in [1]. Recall the standard notation $\subseteq^*$, where for $X, Y \subseteq \omega$, we write $X \subseteq^* Y$ to denote that $|X \setminus Y| < \omega$.

Definition 18 *Let $\mathcal{U}$ be a nonprincipal ultrafilter.*

(1) *$\mathcal{U}$ is* selective *if for every function $f : \omega \to \omega$, there is an $X \in \mathcal{U}$ such that either $f \restriction X$ is constant or $f \restriction X$ is one-to-one.*
(2) *$\mathcal{U}$ is* Ramsey *if for each 2-coloring $f : [\omega]^2 \to 2$, there is an $X \in \mathcal{U}$ such that $f \restriction [X]^2$ takes on exactly one color. This is denoted by $\omega \to (\mathcal{U})^2$.*
(3) *$\mathcal{U}$ is a* p-point *if for every family $\{X_n : n < \omega\} \subseteq \mathcal{U}$ there is an $X \in \mathcal{U}$ such that $X \subseteq^* X_n$ for each $n < \omega$.*
(4) *$\mathcal{U}$ is a* q-point *if for each partition of ω into finite pieces $\{I_n : n < \omega\}$, there is an $X \in \mathcal{U}$ such that $|X \cap I_n| \leq 1$ for each $n < \omega$.*
(5) *$\mathcal{U}$ is* rapid *if for each function $f : \omega \to \omega$, there exists an $X \in \mathcal{U}$ such that $|X \cap f(n)| \leq n$ for each $n < \omega$.*

It is well-known that for ultrafilters on ω, being Ramsey is equivalent to being selective, and that an ultrafilter is Ramsey if and only if it is both a p-point and a q-point. Every q-point is rapid.

Let $(\mathcal{R}, \leq, r)$ be any topological Ramsey space. Recall that a subset $\mathcal{C} \subseteq \mathcal{R}$ is a *filter* on $(\mathcal{R}, \leq)$ if $\mathcal{C}$ is *closed upwards*, meaning that whenever $X \in \mathcal{C}$ and $X \leq Y$, then also $Y \in \mathcal{C}$; and for every pair $X, Y \in \mathcal{C}$, there is a $Z \in \mathcal{C}$ such that $Z \leq X, Y$.

Definition 19 *A filter $\mathcal{C}$ on a topological Ramsey space $\mathcal{R}$ is called* Ramsey for $\mathcal{R}$ *if $\mathcal{C}$ is a maximal filter and for each $n < \omega$ and each $\mathcal{H} \subseteq \mathcal{AR}_n$, there is a member $C \in \mathcal{C}$ such that either $\mathcal{AR}_n|C \subseteq \mathcal{H}$ or else $\mathcal{AR}_n|C \cap \mathcal{H} = \emptyset$.*

Note that a filter which is Ramsey for $\mathcal{R}$ is a maximal filter on $(\mathcal{R}, \leq)$, meaning that for each $X \in \mathcal{R}\setminus\mathcal{C}$, the filter generated by $\mathcal{C} \cup \{X\}$ is all of $\mathcal{R}$.

Fact 20 *Let $\langle \boldsymbol{A}_n : n < \omega\rangle$ be any generating sequence with $1 \leq J < \omega$. Each filter $\mathcal{C}$ which is Ramsey for $\mathcal{R}(\langle \boldsymbol{A}_n : n < \omega\rangle)$ generates an ultrafilter on the base set $\mathcal{AR}_1$, namely the ultrafilter, denoted $\mathcal{U}_{\mathcal{R}}$, generated by the collection $\{\mathcal{AR}_1|C : C \in \mathcal{C}\}$.*

Proof Let $\mathcal{U}$ denote the collection of $\mathcal{G} \subseteq \mathcal{AR}_1$ such that $\mathcal{G} \supseteq \mathcal{AR}_1|C$ for some $C \in \mathcal{C}$. Certainly $\mathcal{U}$ is a filter on $\mathcal{AR}_1$, since $\mathcal{C}$ is a filter on $\mathcal{R}(\langle \boldsymbol{A}_n : n < \omega\rangle)$. To see that $\mathcal{U}$ is an ultrafilter, let $\mathcal{H} \subseteq \mathcal{AR}_1$ be given. Since $\mathcal{C}$ is Ramsey for $\mathcal{R}(\langle \boldsymbol{A}_n : n < \omega\rangle)$, there is a $C \in \mathcal{C}$ such that either $\mathcal{AR}_1|C \subseteq \mathcal{H}$ or else $\mathcal{AR}_1|C \cap \mathcal{H} = \emptyset$. In the first case, $\mathcal{H} \in \mathcal{U}$; in the second case, $\mathcal{AR}_1\setminus\mathcal{H} \in \mathcal{U}$. □

One of the motivations for generating sequences was to provide a construction scheme for ultrafilters which are p-points satisfying some partition relations. At this point, we show how some historic examples of such ultrafilters can be seen to arise as ultrafilters associated with some topological Ramsey space constructed from a generating sequence, thus providing a general framework for such ultrafilters.

Example 21 (A weakly Ramsey, non-Ramsey ultrafilter, [2,17]) In [11] a topological Ramsey space called $\mathcal{R}_1$ was extracted from a forcing of Laflamme which forces a weakly Ramsey ultrafilter which is not Ramsey. That forcing of Laflamme is the same as the example of Baumgartner and Taylor in Theorems 4.8 and 4.9 in [2]. $\mathcal{R}_1$ is exactly $\mathcal{R}(\langle \boldsymbol{A}_n : n < \omega\rangle)$, where each $\boldsymbol{A}_n = \langle n, <\rangle$, the linear order of cardinality n. $\mathcal{R}_1$ is dense in the forcing given by Baumgartner and Taylor. Thus, their ultrafilter can be seen to be generated by the topological Ramsey space $\mathcal{R}_1$.

The next set of examples of ultrafilters which are generated by our topological Ramsey spaces are the n-arrow, not $(n+1)$-arrow ultrafilters of Baumgartner and Taylor.

Definition 22 ([2]) *An ultrafilter $\mathcal{U}$ is* n-arrow *if $3 \leq n < \omega$ and for every function $f : [\omega]^2 \to 2$, either there exists a set $X \in \mathcal{U}$ such that $f([X]^2) = \{0\}$, or else there is a set $Y \in [\omega]^n$ such that $f([Y]^2) = \{1\}$. $\mathcal{U}$ is an* arrow *ultrafilter if $\mathcal{U}$ is n-arrow for each $n \leq 3 < \omega$.*

Theorem 4.11 in [2] of Baumgartner and Taylor shows that for each $2 \leq n < \omega$, there are p-points which are n-arrow but not $(n+1)$-arrow. (By default, every ultrafilter is 2-arrow.) As the ultrafilters of Laflamme [17] with partition relations had led to the formation of new topological Ramsey spaces and their analogues of the Pudlák–Rödl Theorem in [11,12], Todorcevic suggested that these arrow ultrafilters with asymmetric partition relations might lead to interesting new Ramsey-classification theorems. It turns out that the constructions of Baumgartner and Taylor can be thinned to see that there is a generating sequence with associated topological Ramsey space producing their ultrafilters. In fact, our idea of using Fraïssé classes of relational structures to construct topological Ramsey spaces was gleaned from their theorem.

Example 23 (Spaces $\mathcal{A}_n$, generating n-arrow, not $(n+1)$-arrow p-points) For a fixed $n \geq 2$, let $J = 1$ and $\mathcal{K} = \mathcal{K}_0$ denote the Fraïssé class of all finite $(n+1)$-clique-free ordered graphs. By Theorem A of Nešetřil and Rödl [21], $\mathcal{K}$ has the Ramsey property. Choose any generating sequence $\langle \boldsymbol{A}_k : k < \omega \rangle$. One can check, by a proof similar to that given in Theorem 4.11 of [2], that any ultrafilter on $\mathcal{AR}_1$ which is Ramsey for $\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega \rangle)$ is an n-arrow p-point which is not $(n+1)$-arrow.

Let $\mathcal{U}_{\mathcal{A}_n}$ denote any ultrafilter on $\mathcal{AR}_1$ which is Ramsey for $\mathcal{A}_n$. It will follow from Theorem 67 that the initial Rudin-Keisler structure of the p-points Tukey reducible to $\mathcal{U}_{\mathcal{A}_n}$ is exactly that of the collection of isomorphism classes of members of $\mathcal{K}_0$, partially ordered by embedability. Further, Theorem 60 will show that the initial Tukey structure below $\mathcal{U}_{\mathcal{A}_n}$ is exactly a chain of length 2.

Remark In fact, Theorem A in [21] of Nešetřil and Rödl provides a large collection of Fraïssé classes of finite ordered relational structures which omit subobjects which are irreducible. Generating sequences can be taken from any of these, resulting in new topological Ramsey spaces and associated ultrafilters. (See [21] for the relevant definitions.)

The next collection of topological Ramsey spaces we will call *hypercube spaces*, $\mathcal{H}^n$, $1 \leq n < \omega$. The idea for the space $\mathcal{H}^2$ was gleaned from Theorem 9 of Blass [3], where he shows that, assuming Martin's Axiom, there is a p-point with two Rudin-Keisler incomparable p-points Rudin-Keisler reducible to it. The partial ordering he uses has members which are infinite unions of n-squares. That example was enhanced in [10] to show that, assuming CH, there is a p-point with two Tukey-incomparable p-points Tukey reducible to it. A closer look at the partial ordering of Blass reveals inside essentially a product of two copies of the topological Ramsey space $\mathcal{R}_1$ from [11]. Our space $\mathcal{H}^2$ was constructed in order to construct or force a p-point which has initial Tukey structure exactly the Boolean algebra $\mathcal{P}(2)$. The spaces $\mathcal{H}^n$ were then the logical next step in constructing p-points with initial Tukey structure exactly $\mathcal{P}(n)$.

We point out that the space $\mathcal{H}^1$ is exactly the space $\mathcal{R}_1$ in [11].

Remark The space $\mathcal{H}^2$ was investigated in [27]. All the results in this paper pertaining to the space $\mathcal{H}^2$ are due to Trujillo.

Example 24 (Hypercube Spaces $\mathcal{H}^n$, $1 \leq n < \omega$) Fix $1 \leq n < \omega$, and let $J = n$. For each $k < \omega$ and $j \in n$, let $\boldsymbol{A}_{k,j}$ be any linearly ordered set of size $k+1$. Letting $\boldsymbol{A}_k$ denote the sequence $(\boldsymbol{A}_{k,j})_{j \in n}$, we see that $\langle \boldsymbol{A}_k : k < \omega \rangle$ is a generating sequence, where each $\mathcal{K}_j$ is the class of finite linearly ordered sets. Let $\mathcal{H}^n$ denote $\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega \rangle)$. It will follow from Theorem 60 that the initial Tukey structure below $\mathcal{U}_{\mathcal{H}^n}$ is exactly that of the Boolean algebra $\mathcal{P}(n)$.

Many other examples of topological Ramsey spaces are obtained in this manner, simply letting $\mathcal{K}_n$ be a Fraïssé class of finite ordered relational structures with the Ramsey property.

We now look at the most basic example of a topological Ramsey space generated by infinitely many Fraïssé classes. When $J = \omega$, $\mathcal{AR}_1$ no longer suffices as a base for an ultrafilter. In fact, any filter which is Ramsey for this kind of space codes a

Fubini product of the ultrafilters associated with $\mathcal{K}_j$ for each index $j \in \omega$. However, the notion of a filter Ramsey for such a space is still well-defined.

Example 25 (The infinite Hypercube Space $\mathcal{H}^\omega$) Let $J = \omega$. For each $k < \omega$ and $j \in k$, let $\boldsymbol{A}_{k,j}$ be any linearly ordered set of size $k+1$. Letting $\boldsymbol{A}_k$ denote the sequence $(\boldsymbol{A}_{k,j})_{j \in k+1}$, we see that $\langle \boldsymbol{A}_k : k < \omega \rangle$ is a generating sequence for the Fraïssé classes $\mathcal{K}_j$ being the class of finite linearly ordered sets. Let $\mathcal{H}^\omega$ denote $\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega \rangle)$. It will be shown in Theorem 60 that the structure of the Tukey types of p-points Tukey reducible to any filter $\mathcal{C}_{\mathcal{H}^\omega}$ which is Ramsey for $\mathcal{H}^\omega$ is exactly $[\omega]^{<\omega}$. The space $\mathcal{H}^\omega$ is the first example of a topological Ramsey space which has associated filter $\mathcal{C}_{\mathcal{H}^\omega}$ with infinitely many Tukey-incomparable Ramsey ultrafilters Tukey reducible to it.

We point out that, taking $J = \omega$ and each $\mathcal{K}_j$, $j \in \omega$, to be the Fraïssé class of finite ordered $(j+3)$-clique-free graphs, the resulting topological Ramsey space codes the Fubini product seen in Theorem 3.12 in [2] of Baumgartner and Taylor which produces an ultrafilter which is n-arrow for all n.

We conclude this section by showing how the partition properties of ultrafilters Ramsey for some space constructed from a generating sequence can be read off from the Fraïssé classes. Recall the following notation for partition relations. For $k > l$, any $m \geq 2$, and an ultrafilter $\mathcal{U}$,

$$\mathcal{U} \to (\mathcal{U})^m_{k,l} \tag{2}$$

denotes that for any $U \in \mathcal{U}$ and any partition of $[U]^m$ into k pieces, there is a subset $V \subseteq U$ in $\mathcal{U}$ such that $[V]^m$ is contained in at most l pieces of the partition. We shall say that the *Ramsey degree for m-tuples* for $\mathcal{U}$ is l, denoted $R(\mathcal{U}, m) = l$, if $\mathcal{U} \to (\mathcal{U})^m_{k,l}$ for each $k \geq l$, but $\mathcal{U} \not\to (\mathcal{U})^m_{k,l-1}$.

It is straightforward to calculate the Ramsey degrees of ultrafilters Ramsey for topological Ramsey spaces constructed from a generating sequence, given knowledge of the Fraïssé classes used in the construction. For a given Fraïssé class $\mathcal{K}$, for each $s \geq 1$, let $\mathrm{Iso}(\mathcal{K}, s)$ denote the number of isomorphism classes in $\mathcal{K}$ of structures with universe of size s. Let $S(m)$ denote the collection of all finite sequences $\mathbf{s} = \langle s_0, \ldots, s_{l-1} \rangle \in (\omega \setminus \{0\})^{<\omega}$ such that $s_0 + \cdots + s_{l-1} = m$.

Fact 26 *Let $J = 1$, $\mathcal{K}$ be a Fraïssé class of finite ordered relational structures with the Ramsey property, and $\mathcal{U}_\mathcal{K}$ be an ultrafilter Ramsey for $\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega \rangle)$ for some generating sequence for $\mathcal{K}$. Then for each $m \geq 2$,*

$$R(\mathcal{U}_\mathcal{K}, m) = \Sigma_{s \in S(m)} \Pi_{i < |s|} \, \mathrm{Iso}(\mathcal{K}, s_i). \tag{3}$$

Examples 27 For an ultrafilter $\mathcal{U}_{\mathcal{H}^1}$ Ramsey for the space $\mathcal{H}^1$, we have $R(\mathcal{U}_{\mathcal{H}^1}, 2) = 2$, $R(\mathcal{U}_{\mathcal{H}^1}, 3) = 4$, $R(\mathcal{U}_{\mathcal{H}^1}, 4) = 8$, and in general, $R(\mathcal{U}_{\mathcal{H}^1}, m) = 2^{m-1}$.

For an ultrafilter $\mathcal{U}_{\mathcal{A}_2}$ Ramsey for the space $\mathcal{A}_2$, we have $R(\mathcal{U}_{\mathcal{A}_2}, 2) = 3$, $R(\mathcal{U}_{\mathcal{A}_2}, 3) = 12$, and $R(\mathcal{U}_{\mathcal{A}_2}, 4) = 35$. In fact, for each $n \geq 3$, $R(\mathcal{U}_{\mathcal{A}_n}, 2) = 3$, since the only relation is the edge relation. The numbers $R(\mathcal{U}_{\mathcal{A}_n}, m)$ can be calculated from the recursive formula in Fact 26, but as they grow quickly, we leave this to the interested reader.

When $J = 2$, the Ramsey degrees are again calculated from knowledge of the Fraïssé classes $\mathcal{K}_0$ and $\mathcal{K}_1$.

Fact 28 *For $\mathcal{R}$ a topological Ramsey space constructed from a generating sequence for Fraïssé classes $\mathcal{K}_j$, $j \in 2$, letting $\mathcal{U}_{\mathcal{R}}$ be an ultrafilter Ramsey for $\mathcal{R}$, we have*

$$R(\mathcal{U}_{\mathcal{R}}, 2) = 1 + \text{Iso}(\mathcal{K}_0, 2) + \text{Iso}(\mathcal{K}_1, 2) + 2\,\text{Iso}(\mathcal{K}_0, 2) \cdot \text{Iso}(\mathcal{K}_1, 2). \tag{4}$$

The 1 comes from the fact that a pair can come from different blocks; for a pair coming from the same block, $\text{Iso}(\mathcal{K}_0, 2)$ takes care of the case when the pair has the same second dimensional coordinate, $\text{Iso}(\mathcal{K}_1, 2)$ takes care of the case when the pair has the same first dimensional coordinate, and $2\,\text{Iso}(\mathcal{K}_0, 2) \cdot \text{Iso}(\mathcal{K}_1, 2)$ is the number of possible different colors for pairs which are diagonal to each other.

For larger J and m, the Ramsey degrees can be obtained in a similar manner as above. For example, $R(\mathcal{U}_{\mathcal{H}^2}, 3) = 24$. We leave the reader with the following: $R(\mathcal{U}_{\mathcal{H}^2}, 2) = 5$, $R(\mathcal{U}_{\mathcal{H}^3}, 2) = 14$, and we conjecture that in general, $R(\mathcal{U}_{\mathcal{H}^n}, 2) = \frac{3^n-1}{2} + 1$.

5 Canonical equivalence relations for products of structures from Fraïssé classes of finite ordered relational structures

In the main theorem of this section, Theorem 31, we extend the finite Erdős-Rado Theorem 11 to finite products of sets as well as finite products of members of Fraïssé classes of finite ordered relational structures with the Ramsey property and an additional property which we shall call the *Order-prescribed Free Amalgamation Property*, defined below. In particular, this extends the Product Ramsey Theorem 14 from finite colorings to equivalence relations for Fraïssé classes with the aforementioned properties. Theorem 31 will follow from Theorem 32, which gives canonical equivalence relations for blocks from topological Ramsey spaces constructed from generating sequences for these special types of Fraïssé classes. We proceed in this manner for two reasons. First, the strength of topological Ramsey space theory, and in particular the availability of the Abstract Nash-Williams Theorem, greatly streamlines the proof. Second, our desired application of Theorem 31 is in the proof of Theorem 38 in Sect. 6 to find the canonical equivalence relations on fronts for topological Ramsey spaces constructed from a generating sequence.

Recall that $|\boldsymbol{B}|$ denotes the universe of the structure $\boldsymbol{B}$, and $\|\boldsymbol{B}\|$ denotes the cardinality of the universe of $\boldsymbol{B}$. For a structure $\boldsymbol{X}_j \in \mathcal{K}_j$, we shall let $\{x_j^p : p < \|\boldsymbol{X}_j\|\}$ denote the members of the universe $|\boldsymbol{X}_j|$ of $\boldsymbol{X}_j$, enumerated in $<$-increasing order.

Definition 29 (Order-Prescribed Free Amalgamation Property (OPFAP)) *An ordered relational Fraïssé class $\mathcal{K}$ has the* Order-Prescribed Free Amalgamation Property *if the following holds. Suppose $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ are structures in $\mathcal{K}$ with embeddings $e : \boldsymbol{Z} \to \boldsymbol{X}$ and $f : \boldsymbol{Z} \to \boldsymbol{Y}$. Let $K = \|\boldsymbol{X}\|$, $L = \|\boldsymbol{Y}\|$, and $M = \|\boldsymbol{Z}\|$. Let $\{x^k : k \in K\}$ denote $|\boldsymbol{X}|$ and $\{y^l : l \in L\}$ denote $|\boldsymbol{Y}|$, the universes of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Let $K' = \{k'_m : m \in M\} \subseteq K$ and $L' = \{l'_m : m \in M\} \subseteq L$ be the subsets such that $\boldsymbol{X} \restriction K' = e(\boldsymbol{Z})$ and $\boldsymbol{Y} \restriction L' = f(\boldsymbol{Z})$.*

Let $\boldsymbol{\rho} : K \times L \to \{<, =, >\}$ be any function such that

(a) *For each $m \in M$, $\boldsymbol{\rho}(k'_m, l'_m) = =$;*
(b) *For each $m \in M$, $k < k'_m$ and $l > l'_m$ implies $\boldsymbol{\rho}(k, l) = <$; and $k > k'_m$ and $l < l'_m$ implies $\boldsymbol{\rho}(k, l) = >$;*
(c) *For all $(k, l) \in (K \setminus K') \times (L \setminus L')$, $\boldsymbol{\rho}(k, l) \neq =$;*
(d) *$\boldsymbol{\rho}(k, l) = <$ implies for all $l' > l$ and $k' < k$, $\boldsymbol{\rho}(k', l') = <$; and $\boldsymbol{\rho}(k, l) = >$ implies for all $l' < l$ and $k' > k$, $\boldsymbol{\rho}(k', l') = >$.*

Then there is a free amalgamation $(g, h, \boldsymbol{W})$ of $(\boldsymbol{Z}, e, \boldsymbol{X}, f, \boldsymbol{Y})$ and there is a function $\sigma : K + L \to \|\boldsymbol{W}\|$ such that the following hold:

(1) *$\sigma \restriction K$ and $\sigma \restriction [K, K + L)$ are strictly increasing;*
(2) *$\boldsymbol{W} \restriction \sigma''K = g(\boldsymbol{X})$ and $\boldsymbol{W} \restriction \sigma''[K, K + L) = h(\boldsymbol{Y})$;*
(3) *For all $m \in M$, $\sigma(k'_m) = \sigma(K + l'_m)$, and $\boldsymbol{W} \restriction \sigma''\{k'_m : m \in M\} = \boldsymbol{W} \restriction \sigma''\{K + l'_m : m \in M\} = g \circ e(\boldsymbol{Z}) = h \circ f(\boldsymbol{Z}) \cong \boldsymbol{Z}$;*
(4) *For all $(k, l) \in K \times L$, $w^{\sigma(k)}\, \boldsymbol{\rho}\, w^{\sigma(K+l)}$.*

Hence, $\boldsymbol{W}$ contains copies of $\boldsymbol{X}$ and $\boldsymbol{Y}$ which appear as substructures of $\boldsymbol{W}$ in the order prescribed by $\boldsymbol{\rho}$.

In words, the Order-Prescribed Free Amalgamation Property says that given any structure $\boldsymbol{Z}$ appearing as a substructure of both $\boldsymbol{X}$ and $\boldsymbol{Y}$, one can find a strong amalgamation $\boldsymbol{W}$ of $\boldsymbol{X}$ and $\boldsymbol{Y}$ so that the members of the universes of the copies of $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\boldsymbol{W}$ lying between the members of the universe of the copy of $\boldsymbol{Z}$ can lie in any order which we prescribed ahead of time, and the only relations between members of the copies of $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\boldsymbol{W}$ are those in $\boldsymbol{Z}$, that is, the amalgamation is free.

Remark Note that (3) in Definition 29 implies that there are no transitive relations on $\mathcal{K}$. Thus, any Fraïssé class which has a transitive relation does not satisfy the OPFAP. We point out that the classes of ordered finite graphs, ordered finite K_n-free graphs, and more generally, the classes of ordered set-systems omitting some collection of irreducible structures (see [21]) all satisfy the OPFAP.

Definition 30 *Let $\mathcal{K}_j$, $j \in J < \omega$ be Fraïssé classes of finite ordered relational structures with the Ramsey property and the OPFAP. For each $j \in J$, let $\boldsymbol{A}_j, \boldsymbol{B}_j \in \mathcal{K}_j$ such that $\boldsymbol{A}_j \leq \boldsymbol{B}_j$. Given a subset $I_j \subseteq \|\boldsymbol{A}_j\|$ and $\boldsymbol{X}_j, \boldsymbol{Y}_j \in \binom{\boldsymbol{B}_j}{\boldsymbol{A}_j}$, we write $|\boldsymbol{X}_j|\, E_{I_j}\, |\boldsymbol{Y}_j|$ if and only if for all $i \in I_j$, $x^i_j = y^i_j$.*

An equivalence relation E on $\binom{(\boldsymbol{B}_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$ is canonical *if and only if for each $j \in J$, there is a set $I_j \subseteq \|\boldsymbol{A}_j\|$ such that for all $(\boldsymbol{X}_j)_{j\in J}, (\boldsymbol{Y}_j)_{j\in J} \in \binom{(\boldsymbol{B}_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$,*

$$(\boldsymbol{X}_j)_{j\in J}\, E\, (\boldsymbol{Y}_j)_{j\in J} \longleftrightarrow \forall j \in J,\ |\boldsymbol{X}_j|\, E_{I_j}\, |\boldsymbol{Y}_j|. \tag{5}$$

When E is canonical, given by E_{I_j}, $j \in J$, then we shall write $E = E_{(I_j)_{j\in J}}$.

Theorem 31 *Let $\mathcal{K}_j$, $j \in J < \omega$, be Fraïssé classes of ordered relational structures with the Ramsey property and the OPFAP. For each $j \in J$, let $\boldsymbol{A}_j, \boldsymbol{B}_j \in \mathcal{K}_j$ be such that $\boldsymbol{A}_j \leq \boldsymbol{B}_j$. Then for each $j \in J$, there is a $\boldsymbol{C}_j \in \mathcal{K}_j$ such that for each equivalence* *relation E on $\binom{(\boldsymbol{C}_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$, there is a sequence $(\boldsymbol{B}'_j)_{j\in J} \in \binom{(\boldsymbol{C}_j)_{j\in J}}{(\boldsymbol{B}_j)_{j\in J}}$ such that E restricted to $\binom{(\boldsymbol{B}'_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$ is canonical.*

Theorem 31 will follow immediately from the next theorem.

Theorem 32 *Let $\langle \boldsymbol{A}_k : k < \omega\rangle$ be a generating sequence associated to some Fraïssé classes of finite ordered relational structures $\mathcal{K}_j$, $j \in J$, each satisfying the Ramsey property and the OPFAP. Let $n < \omega$ and $L \subseteq J_n$ be given, and let E be an equivalence relation on $\bigcup_{k\geq n}\binom{(\boldsymbol{A}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}}$ such that $E \subseteq \bigcup_{k\geq n}\binom{(\boldsymbol{A}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}} \times \binom{(\boldsymbol{A}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}}$. Then there is a $C \in \mathcal{R}(\langle \boldsymbol{A}_k : k < \omega\rangle)$ and there are index sets $I_j \subseteq \|\boldsymbol{A}_{n,j}\|$ such that for all $k \geq n$, $E = E_{(I_j)_{j\in L}}$ when restricted to $\binom{(\boldsymbol{C}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}}$. That is, for each $k \geq n$, and each pair $(\boldsymbol{X}_{n,j})_{j\in L}, (\boldsymbol{Y}_{n,j})_{j\in L} \in \binom{(\boldsymbol{C}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}}$,*

$$\big((\boldsymbol{X}_{n,j})_{j\in L}\ E\ (\boldsymbol{Y}_{n,j})_{j\in L} \longleftrightarrow \forall j \in L,\ |\boldsymbol{X}_{n,j}|\ E_{I_j}\ |\boldsymbol{Y}_{n,j}|\big)\,.$$

Proof Let $J \leq \omega$ and $\mathcal{K}_j$, $j \in J$, be a collection of Fraïssé classes of finite ordered relational structures with the Ramsey property and the Order-Prescribed Free Amalgamation Property. Let $\langle \boldsymbol{A}_k : k < \omega\rangle$ be a generating sequence associated with the $\mathcal{K}_j$, $j \in J$, and let $\mathcal{R}$ denote the topological Ramsey space $\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega\rangle)$. Recall that $J_n = J$ if $J < \omega$, and $J_n = n$ if $J = \omega$.

Before beginning the inductive proof, we establish some terminology and notation, and Lemma 33 below. Given $n < \omega$, for each $j \in J_n$ let K_j denote $\|\boldsymbol{A}_{n,j}\|$, the cardinality of the universe of the structure $\boldsymbol{A}_{n,j}$. For a given structure $\boldsymbol{X} \in \mathcal{K}_j$, let $\{x^0, \dots, x^{\|\boldsymbol{X}\|-1}\}$ denote $|\boldsymbol{X}|$, the universe of $\boldsymbol{X}$, enumerated in increasing order. For $M \subseteq \|X\|$, let $\boldsymbol{X} \restriction M$ denote the substructure of $\boldsymbol{X}$ on universe $\{x^k : k \in M\}$. For each $j \in J_n$, let $\mathrm{Amalg}(n, j)$ denote the collection of all $\boldsymbol{X} \in \mathcal{K}_j$ such that $\boldsymbol{X}$ is an amalgamation of two copies of $\boldsymbol{A}_{n,j}$. By this we mean precisely that there are set of indices $M_0, M_1 \subseteq \|\boldsymbol{X}\|$ such that M_0 and M_1 each have cardinality K_j, $M_0 \cup M_1 = \|\boldsymbol{X}\|$, and $\boldsymbol{X} \restriction M_0 \cong \boldsymbol{X} \restriction M_1 \cong \boldsymbol{A}_{n,j}$.

By the definition of a generating sequence, given $n < \omega$ there is an $m > n$ such that for each $j \in J_n$, every structure $\boldsymbol{X} \in \mathrm{Amalg}(n, j)$ embeds into $\boldsymbol{A}_{m,j}$. Define $\mathcal{I}_j$ to be the collection of functions $\iota_j : 2K_j \to \|\boldsymbol{A}_{m,j}\|$ such that $\iota_j \restriction K_j$ and $\iota_j \restriction [K_j, 2K_j)$ are strictly increasing, the substructure $\boldsymbol{A}_{m,j} \restriction \iota''2K_j$ is in $\mathrm{Amalg}(n, j)$, and moreover, $\boldsymbol{A}_{m,j} \restriction \iota''K_j \cong \boldsymbol{A}_{m,j} \restriction \iota''[K_j, 2K_j) \cong \boldsymbol{A}_{n,j}$. For each $\iota_j \in \mathcal{I}_j$ and $\boldsymbol{X} \cong \boldsymbol{A}_{m,j}$, fix the notation

$$\begin{aligned}\iota_j(\boldsymbol{X}) &:= \big(\boldsymbol{X} \restriction \iota''K_j, \boldsymbol{X} \restriction \iota''[K_j, 2K_j)\big)\\ &= \left(\left\{x^{\iota_j(0)}, \dots, x^{\iota_j(K_j-1)}\right\}, \left\{x^{\iota_j(K_j)}, \dots, x^{\iota_j(2K_j-1)}\right\}\right),\end{aligned}$$

the pair of structures in $\binom{\boldsymbol{X}}{\boldsymbol{A}_{n,j}}$ determined by ι_j.

Throughout the poof of this theorem, given any structure $\boldsymbol{D}$ which embeds $\boldsymbol{A}_{n,j}$, for any $\boldsymbol{X}, \boldsymbol{Y} \in \binom{\boldsymbol{D}}{\boldsymbol{A}_{n,j}}$, the pair $(\boldsymbol{X}, \boldsymbol{Y})$ is considered both as an ordered pair of structures isomorphic to $\boldsymbol{A}_{n,j}$ as well as the substructure of $\boldsymbol{D} \restriction (|\boldsymbol{X}| \cup |\boldsymbol{Y}|)$ with all inherited relations.

Claim 1 *Let $j \in J_n$. There is a structure $\boldsymbol{B} \in \mathcal{K}_j$ with a substructure $\boldsymbol{C} \in \binom{\boldsymbol{B}}{\boldsymbol{A}_{m,j}}$ such that for each $\iota \in \mathcal{I}_j$, for each $\tau \in \mathcal{I}_j$ such that $\tau(\boldsymbol{A}_{m,j}) \cong \iota(\boldsymbol{C})$ there is a $\boldsymbol{V} \in \binom{\boldsymbol{B}}{\boldsymbol{A}_{m,j}}$ such that $\tau(\boldsymbol{V}) = \iota(\boldsymbol{C})$.*

Proof Let $p = |\mathcal{I}_j| - 1$ and enumerate $\mathcal{I}_j$ as $\langle \iota^i : i \le p\rangle$. The proof proceeds by amalgamation in p stages, each stage $i \le p$ proceeding inductively by amalgamating to obtaining a $\boldsymbol{B}^i$ which satisfies the claim for the structure $\iota_i(\boldsymbol{A}_{m,j})$.

Let $\boldsymbol{W}^0$ denote the substructure $\iota^0(\boldsymbol{A}_{m,j})$. Let $\mathcal{I}^0$ denote the set of $\tau \in \mathcal{I}_j$ such that $\tau(\boldsymbol{A}_{m,j}) \cong \boldsymbol{W}^0$, and enumerate $\mathcal{I}^0$ as $\langle \tau^{0,k} : k \le q^0\rangle$. Let $e^{0,0} : \boldsymbol{W}^0 \to \boldsymbol{A}_{m,j}$ be the identity injection on $\boldsymbol{W}^0$, so that $e^{0,0}(\boldsymbol{W}^0) = \iota^0(\boldsymbol{A}_{m,j})$. Let $f^{0,0} : \boldsymbol{W}^0 \to \boldsymbol{V}^{0,0}$ be an embedding of $\boldsymbol{W}^0$ into a copy $\boldsymbol{V}^{0,0}$ of $\boldsymbol{A}_{m,j}$ such that $f^{0,0}(\boldsymbol{W}^0) = \tau^{0,0}(\boldsymbol{V}^{0,0})$. Let $(g^{0,0}, h^{0,0}, \boldsymbol{B}^{0,0})$ be a free amalgamation of $(\boldsymbol{W}^0, e^{0,0}, \boldsymbol{A}_{m,j}, f^{0,0}, \boldsymbol{V}^{0,0})$, and let $e^{0,1} = g^{0,0} \circ e^{0,0}$. Thus, $e^{0,1} : \boldsymbol{W}^0 \to \boldsymbol{B}^{0,0}$. Let $f^{0,1} : \boldsymbol{W}^0 \to \boldsymbol{V}^{0,1}$ be an embedding of $\boldsymbol{W}^0$ into a copy $\boldsymbol{V}^{0,1}$ of $\boldsymbol{A}_{m,j}$ such that $f^{0,1}(\boldsymbol{W}^0) = \tau^{0,1}(\boldsymbol{V}^{0,1})$. Let $(g^{0,1}, h^{0,1}, \boldsymbol{B}^{0,1})$ be a free amalgamation of $(\boldsymbol{W}^0, e^{0,1}, \boldsymbol{B}^{0,0}, f^{0,1}, \boldsymbol{V}^{0,1})$, and let $e^{0,2} = g^{0,1} \circ e^{0,1}$, so that $e^{0,2} : \boldsymbol{W}^0 \to \boldsymbol{B}^{0,1}$.

Given $e^{0,k+1} : \boldsymbol{W}^0 \to \boldsymbol{B}^{0,k}$, let $f^{0,k+1} : \boldsymbol{W}^0 \to \boldsymbol{V}^{0,k+1}$ be an embedding of $\boldsymbol{W}^0$ into a copy $\boldsymbol{V}^{0,k+1}$ of $\boldsymbol{A}_{m,j}$ such that $f^{0,k+1}(\boldsymbol{W}^0) = \tau^{0,k+1}(\boldsymbol{V}^{0,k+1})$. Let $(g^{0,k+1}, h^{0,k+1}, \boldsymbol{B}^{0,k+1})$ be a free amalgamation of $(\boldsymbol{W}^0, e^{0,k+1}, \boldsymbol{B}^{0,k}, f^{0,k_1}, \boldsymbol{V}^{0,k+1})$, and let $e^{0,k+2} = g^{0,k+1} \circ e^{0,k+1}$, so that $e^{0,k+2} : \boldsymbol{W}^0 \to \boldsymbol{B}^{0,k+1}$. At the end of the q^0 many stages of the construction, let $h^0 = h^{0,q^0} \circ \cdots \circ h^{0,0}$ and let $\boldsymbol{B}^0 = \boldsymbol{B}^{0,q^0}$, so that h^0 embeds the original copy of $\boldsymbol{A}_{m,j}$ into $\boldsymbol{B}^0$. This concludes the 0-th stage of constructing $\boldsymbol{B}$.

For the i-th stage, suppose that $0 < i \le p$ and $h^{i-1} : \boldsymbol{A}_{m,j} \to \boldsymbol{B}^{i-1}$ are given. Let $\mathcal{I}^i$ denote the set of $\tau \in \mathcal{I}_j$ such that $\tau(\boldsymbol{A}_{m,j}) \cong \boldsymbol{W}^i := \iota^i(\boldsymbol{A}_{m,j})$, and enumerate these as $\langle \tau^{i,k} : k \le q^i\rangle$. Let $e^{i,0} : \boldsymbol{W}^i \to \boldsymbol{B}^{i-1}$ be the embedding such that $e^{i,0}(\boldsymbol{W}^i) = \iota^i(h^{i-1}(\boldsymbol{A}_{m,j}))$. Let $f^{i,0} : \boldsymbol{W}^i \to \boldsymbol{V}^{i,0}$ be an embedding of $\boldsymbol{W}^i$ into a copy $\boldsymbol{V}^{i,0}$ of $\boldsymbol{A}_{m,j}$ such that $f^{i,0}(\boldsymbol{W}^i) = \tau^{i,0}(\boldsymbol{V}^{i,0})$. Let $(g^{i,0}, h^{i,0}, \boldsymbol{B}^{i,0})$ be a free amalgamation of $(\boldsymbol{W}^i, e^{i,0}, \boldsymbol{B}_{i-1}, f^{i,0}, \boldsymbol{V}^{i,0})$, and let $e^{i,1} = g^{i,0} \circ e^{i,0}$.

For $0 < k \le q^i$ suppose $e^{i,k} : \boldsymbol{W}^i \to \boldsymbol{B}^{i,k-1}$ is given. Let $f^{i,k} : \boldsymbol{W}^i \to \boldsymbol{V}^{i,k}$ be an embedding of $\boldsymbol{W}^i$ into a copy $\boldsymbol{V}^{i,k}$ of $\boldsymbol{A}_{m,j}$ such that $f^{i,k}(\boldsymbol{W}^i) = \tau^{i,k}(\boldsymbol{V}^{i,k})$. Let $(g^{i,k}, h^{i,k}, \boldsymbol{B}^{i,k})$ be a free amalgamation of $(\boldsymbol{W}^i, e^{i,k}, \boldsymbol{B}^{i,k-1}, f^{i,k}, \boldsymbol{V}^{i,k})$, and let $e^{i,k+1} = g^{i,k} \circ e^{i,k}$. At the end of the q^i many stages of the construction, let $\boldsymbol{B}^i = \boldsymbol{B}^{i,q^i}$ and $h^i = h^{i,q^i} \circ \cdots \circ h^{i,0}$. This concludes the i-th stage.

Repeating the procedure for all $i \le p$, we obtain a structure $\boldsymbol{B} := \boldsymbol{B}^p$ satisfying the claim. □

Claim 2 *Let $B \in \mathcal{R}$, $n < \omega$ and $L \subseteq J_n$ be given. There is a $C \le B$ such that for all $k \ge m$, for each pair $(\boldsymbol{X}_j)_{j\in L}, (\boldsymbol{Y}_j)_{j\in L} \in \binom{(\boldsymbol{C}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}}$ and for each $(\iota_j : j \in L) \in \prod_{j\in L} \mathcal{I}_j$ such that for each $j \in L$, $\iota_j(\boldsymbol{A}_{m,j}) \cong (\boldsymbol{X}_j, \boldsymbol{Y}_j)$, there is a $(\boldsymbol{Z}_j)_{j\in L} \in \binom{(\boldsymbol{B}_{k',j})_{j\in L}}{(\boldsymbol{A}_{m,j})_{j\in L}}$ such that each $\iota_j(\boldsymbol{Z}_j) = (\boldsymbol{X}_j, \boldsymbol{Y}_j)$, where k' is such that $\boldsymbol{C}_{k,j}$ is a substructure of $\mathbf{B}_{k',j}$.*

Proof By Claim 1, for each $p \geq m$ and each $j \in L$, there is a structure $\boldsymbol{B}^*_{p,j} \in \mathcal{K}_j$ containing a substructure $\boldsymbol{C}^*_{p,j} \in \binom{\boldsymbol{B}^*_{p,j}}{\boldsymbol{A}_{p,j}}$ with the following property: Given $\boldsymbol{U}_j \in \binom{\boldsymbol{C}^*_{p,j}}{\boldsymbol{A}_{m,j}}$ and $\iota_j \in \mathcal{I}_j$, for each $\tau_j \in \mathcal{I}_j$ such that $\tau_j(\boldsymbol{A}_{m,j}) \cong (\boldsymbol{X}_j, \boldsymbol{Y}_j)$, there is a $\boldsymbol{V}_j \in \binom{\boldsymbol{B}^*_{p,j}}{\boldsymbol{A}_{m,j}}$ such that $\tau_j(\boldsymbol{V}_j) = \iota_j(\boldsymbol{U}_j)$. Since for a generating sequence, each structure in $\mathcal{K}_j$ embeds into all but finitely many $\boldsymbol{A}_{k,j}$, there is a subsequence $(k_p)_{p \geq m}$ such that each $\boldsymbol{B}^*_{p,j}$ embeds as a substructure of $\boldsymbol{A}_{k_p,j}$.

Thinning through this subsequence, for each $p \geq m$, take $\boldsymbol{C}_{p,j}$ to be a substructure of $\boldsymbol{B}_{k_p,j}$ isomorphic to $\boldsymbol{A}_{p,j}$ satisfying Claim 1, and let $\boldsymbol{C}_{p,j} = \boldsymbol{B}_{p,j}$ for each $p < m$. Then we obtain a $C \leq B$ which satisfies the claim. □

For fixed n and $L \subseteq J_n$, we shall let $\mathcal{I}$ denote the set of all sequences $(\iota_j)_{j \in L} \in \prod_{j \in L} \mathcal{I}_j$. Given a sequence $\iota = (\iota_j : j \in L) \in \mathcal{I}$ and $X(m) \in \mathcal{R}(m)$, fix the notation

$$\iota(X(m)) := \left(\left(\boldsymbol{X}_{m,j} \restriction \iota''_j K_{n,j} \right)_{j \in L}, \left(\boldsymbol{X}_{m,j} \restriction \iota''_j [K_{n,j}, 2K_{n,j}) \right)_{j \in L} \right).$$

Thus, $\iota(X(m))$ is a pair of sequences of structures, each sequence of which is isomorphic to $(\boldsymbol{A}_{n,j})_{j \in L}$. Moreover, for each $j \in L$, the pair $(\boldsymbol{X}_{m,j} \restriction \iota''_j K_{n,j}, \boldsymbol{X}_{m,j} \restriction \iota''_j [K_{n,j}, 2K_{n,j}))$ also determines a substructure of $\boldsymbol{X}_{m,j}$ which is an amalgamation of two copies of $\boldsymbol{A}_{n,j}$.

Lemma 33 *Let $n < \omega$ and $L \subseteq J_n$ be given. Let E be an equivalence relation on $\bigcup_{k \geq n} \binom{(\boldsymbol{A}_{k,j})_{j \in L}}{(\boldsymbol{A}_{n,j})_{j \in L}} \times \binom{(\boldsymbol{A}_{k,j})_{j \in L}}{(\boldsymbol{A}_{n,j})_{j \in L}}$. Let m be large enough that for each $j \in L$, all members of* Amalg(n, j) *embed into $\boldsymbol{A}_{m,j}$. Then there are $C \leq B \in \mathcal{R}$ and a subset $\mathcal{I}' \subseteq \mathcal{I}$ such that $C \leq B$ satisfy Claim 2 and for all $k \geq n$ and all $X(n), Y(n) \in \mathcal{R}(n)|C(k)$,*

$$\begin{aligned} &\left(\boldsymbol{X}_{n,j}\right)_{j \in L} \; E \; (\boldsymbol{Y}_{n,j})_{j \in L} \longleftrightarrow \exists U(m) \in \mathcal{R}(m)|B \;\; \exists \iota \in \mathcal{I}' \;\; \iota(U(m)) \\ &\quad = ((\boldsymbol{X}_{n,j})_{j \in L}, (\boldsymbol{Y}_{n,j})_{j \in L})). \end{aligned} \tag{6}$$

Proof For each $\iota \in \mathcal{I}$, define

$$\mathcal{H}_\iota = \{r_{m+1}(X) : X \in \mathcal{R} \text{ and } E(\iota(X(m)))\}.$$

Each $\mathcal{H}_\iota$ is a subset of the Nash-Williams family $\mathcal{AR}_{m+1}$. Hence, by the Abstract Nash-Williams Theorem, there is a $B \in \mathcal{R}$ which is homogeneous for $\mathcal{H}_\iota$, for all $\iota \in \mathcal{I}$. That is, for each $\iota \in \mathcal{I}$, either $\mathcal{AR}_{m+1}|B \subseteq \mathcal{H}_\iota$ or else $\mathcal{AR}_{m+1}|B \cap \mathcal{H}_\iota = \emptyset$. Let $\mathcal{I}' = \{\iota \in \mathcal{I} : E(\iota(B(m)))\}$. Finally, take $C \leq B$ satisfying the conclusion of Claim 2 for each $j \in L$. □

We will prove the following statement by induction on $M \geq 1$: Given any n such that $J_n \geq M$, $L \in [J_n]^M$, and an equivalence relation E on $\bigcup_{k \geq n} \binom{(\boldsymbol{A}_{k,j})_{j \in L}}{(\boldsymbol{A}_{n,j})_{j \in L}} \times \binom{(\boldsymbol{A}_{k,j})_{j \in L}}{(\boldsymbol{A}_{n,j})_{j \in L}}$, there is a $C \in \mathcal{R}$ and there are index sets $I_j \subseteq K_j$ such that for all $k \geq n$ and all $(\boldsymbol{X}_{n,j})_{j \in L}, (\boldsymbol{Y}_n)_{j \in L} \in \binom{(\boldsymbol{C}_{k,j})_{j \in L}}{(\boldsymbol{A}_{n,j})_{j \in L}}$,

$$(\boldsymbol{X}_{n,j})_{j\in L}\ E\ (\boldsymbol{Y}_{n,j})_{j\in L} \text{ if and only if } \forall j \in L,\ |\boldsymbol{X}_{n,j}|\ E_{I_j}\ |\boldsymbol{Y}_{n,j}|.$$

Base Case. $M = 1$. Let $n < \omega$, $j \in J_n$, and $L = \{j\}$. Let E be an equivalence relation such that $E \subseteq \bigcup_{k\geq n} \binom{(\boldsymbol{A}_{k,j})}{(\boldsymbol{A}_{n,j})} \times \binom{(\boldsymbol{A}_{k,j})}{(\boldsymbol{A}_{n,j})}$. Let $C \leq B \in \mathcal{R}$ and $\mathcal{I}'_j$ satisfy Lemma 33. Define

$$I_j = \{i \in K_j : \forall \iota \in \mathcal{I}',\ \iota_j(i) = \iota_j(K_j + i)\}.$$

Since each $\iota \in \mathcal{I}'_j$ is a sequence consisting of only a single entry, (ι_j), we shall abuse notation for the base case and use ι in place of ι_j. We make the convention that for each $k < \omega$, k' denotes the number such that $\boldsymbol{C}_{k,j}$ is a substructure of $\boldsymbol{B}_{k',j}$.

Claim 3 *If $\iota \in \mathcal{I}'_j$, $\tau \in \mathcal{I}_j$, and $\tau(\boldsymbol{A}_{m,j}) \cong \iota(\boldsymbol{A}_{m,j})$, then $\tau \in \mathcal{I}'_j$.*

Proof Let ι and τ be as in the hypothesis. Let $(\boldsymbol{X}, \boldsymbol{Y}) = \iota(\boldsymbol{C}_{m,j})$. By Claim 2, there is an $m' \geq m$ and a $\boldsymbol{V} \in \binom{\boldsymbol{B}_{m',j}}{\boldsymbol{A}_{m,j}}$ such that $\tau(\boldsymbol{V}) = (\boldsymbol{X}, \boldsymbol{Y})$. Since $\iota \in \mathcal{I}'_j$, Lemma 33 implies that $\boldsymbol{X}\ E\ \boldsymbol{Y}$. Therefore, by Lemma 33, τ is also in $\mathcal{I}'_j$. □

Claim 4 *Let $i \in K_j \backslash I_j$ For each $l \geq m$ there are $\boldsymbol{X}, \boldsymbol{Y} \in \binom{\boldsymbol{C}_{l,j}}{\boldsymbol{A}_{n,j}}$ such that for each $k \in K_j\backslash\{i\}$, $x^k = y^k$, $x^i \neq y^i$, and $\boldsymbol{C}_{l,j} \restriction (|\boldsymbol{X}| \cup |\boldsymbol{Y}|)$ is a free amalgamation of $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let ι be any map in $\mathcal{I}_j$ such that $\iota(\boldsymbol{A}_{m,j}) \cong (\boldsymbol{X}, \boldsymbol{Y})$. Then $\iota \in \mathcal{I}'_j$.*

Proof First we prove a general fact. Let $\sigma \in \mathcal{I}_j$ and $i \in K_j$ be such that $\sigma(i) < \sigma(K_j + i)$. Let $\boldsymbol{V} = (\boldsymbol{X}, \boldsymbol{Y}) = \sigma(\boldsymbol{A}_{m,j})$. Let $k \in \|\boldsymbol{V}\|$ be such that $v^k = x^i$. Take another copy $\boldsymbol{W} = (\boldsymbol{Z}, \boldsymbol{Y}') \cong \sigma(\boldsymbol{A}_{m,j})$. Let $e : \boldsymbol{Y} \to \boldsymbol{V}$ be the identity embedding and $f : \boldsymbol{Y} \to \boldsymbol{W}$ be such that $f(\boldsymbol{Y})$ equals $\boldsymbol{Y}'$. By the OPFAP, we may freely amalgamate $(\boldsymbol{Y}, e, \boldsymbol{V}, f, \boldsymbol{W})$ to some $(g, h, \boldsymbol{U})$ so that the following hold: $\boldsymbol{V} \cong \boldsymbol{U} \restriction (\|\boldsymbol{U}\|\backslash\{k\})$ and $\boldsymbol{W} \cong \boldsymbol{U} \restriction (\|\boldsymbol{U}\|\backslash\{k+1\})$. In words, $\boldsymbol{U}$ consists exactly of copies of the substructures $(\boldsymbol{X}, \boldsymbol{Y})$ and $(\boldsymbol{Z}, \boldsymbol{Y}')$ where the copies of $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ coincide, and the copies of $\boldsymbol{X}$ and $\boldsymbol{Z}$ in $\boldsymbol{W}$ differ only on their i-th coordinates. Thus, by an argument similar to Claim 2, possibly thinning C again, we may assume that for each $\sigma \in \mathcal{I}_j$ and $i \in K_j$ such that $\sigma(i) < \sigma(K_j + i)$ and $(\boldsymbol{X}, \boldsymbol{Y}) = \sigma(\boldsymbol{C}_{m,j})$, there are substructures $\boldsymbol{Z}$ and $\boldsymbol{W}$ as above coming from $\boldsymbol{B}_{m',j}$. That is, there is a z^k in $\boldsymbol{B}_{m',j}$ so that the substructure of $\boldsymbol{B}_{m',j}$ restricted to universe of $\sigma(\boldsymbol{C}_{m,j}) \cup \{z^k\}$ is isomorphic to $\boldsymbol{W}$.

To prove the claim, first note that since i is in $K_j\backslash I_j$, there is a $\sigma \in \mathcal{I}'_j$ such that $\sigma(i) \neq \sigma(K_j + i)$. Let $(\boldsymbol{X}, \boldsymbol{Y}) = \sigma(\boldsymbol{C}_{m,j})$. Since σ is in $\mathcal{I}'_j$, it follows that $\boldsymbol{X}\ E\ \boldsymbol{Y}$, by Lemma 33. Without loss of generality, assume that $x^i < y^i$. By the previous paragraph, there are structures $\boldsymbol{Z} \in \binom{\boldsymbol{B}_{m',j}}{\boldsymbol{A}_{n,j}}$ and $\boldsymbol{W} \in \binom{\boldsymbol{B}_{m',j}}{\boldsymbol{A}_{m,j}}$ such that

(1) for each $k \in K_j\backslash\{i\}$, $z^k = x^k$,
(2) $z^i < x^i$,
(3) $\boldsymbol{B}_{m',j} \restriction (|\boldsymbol{X}| \cup |\boldsymbol{Z}|)$ is the free amalgamation of $\boldsymbol{X}$ and $\boldsymbol{Z}$, and
(4) $\sigma(\boldsymbol{W}) = (\boldsymbol{Z}, \boldsymbol{Y})$.

Since σ is in $\mathcal{I}'_j$, it follows that $\boldsymbol{Z}\ E\ \boldsymbol{Y}$. Hence, $\boldsymbol{X}\ E\ \boldsymbol{Z}$.

Now let $\iota \in \mathcal{I}_j$ be any map such that $\iota(\boldsymbol{A}_{m,j}) \cong (\boldsymbol{X}, \boldsymbol{Z})$; that is, the free amalgamation of two copies of $\boldsymbol{A}_{n,j}$ where only their i-th coordinates differ. Then ι must be in $\mathcal{I}'_j$ by Lemma 33, since $\boldsymbol{X}\ E\ \boldsymbol{Z}$ and there is a $\boldsymbol{D} \in \binom{\boldsymbol{B}_{m',j}}{\boldsymbol{A}_{m,j}}$ such that $\iota(\boldsymbol{D}) = (\boldsymbol{X}, \boldsymbol{Z})$. □

Claim 5 *Let $\iota, \tau \in \mathcal{I}'_j$ and $k \geq m$ be given. Suppose there are $\boldsymbol{V}, \boldsymbol{W} \in \binom{\boldsymbol{B}_{k',j}}{\boldsymbol{A}_{m,j}}$ such that $\iota(\boldsymbol{V}) = (\boldsymbol{X}, \boldsymbol{Y})$ and $\tau(\boldsymbol{W}) = (\boldsymbol{X}, \boldsymbol{Z})$, where $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \binom{\boldsymbol{C}_{k,j}}{\boldsymbol{A}_{n,j}}$. Then for each $\sigma \in \mathcal{I}_j$ such that there is a $\boldsymbol{U} \in \binom{\boldsymbol{B}_{k',j}}{\boldsymbol{A}_{m,j}}$ such that $\sigma(\boldsymbol{U}) = (\boldsymbol{Y}, \boldsymbol{Z})$, σ is in $\mathcal{I}'_j$.*

Likewise, if there are $\boldsymbol{V}, \boldsymbol{W} \in \binom{\boldsymbol{B}_{k',j}}{\boldsymbol{A}_{m,j}}$ such that $\iota(\boldsymbol{V}) = (\boldsymbol{X}, \boldsymbol{Y})$ and $\tau(\boldsymbol{W}) = (\boldsymbol{Y}, \boldsymbol{Z})$, then for each $\sigma \in \mathcal{I}_j$ for which there is a $\boldsymbol{U} \in \binom{\boldsymbol{B}_{k',j}}{\boldsymbol{A}_{m,j}}$ such that $\sigma(\boldsymbol{U}) = (\boldsymbol{X}, \boldsymbol{Z})$, σ is in $\mathcal{I}'_j$.

Proof The proof is immediate from Lemma 33, the definition of $\mathcal{I}'_j$ and the fact that E is an equivalence relation. □

Our strategy at this point is to find an $\eta \in \mathcal{I}'_j$ such that for each interval between two points of $\boldsymbol{I}_j$, for $(\boldsymbol{X}, \boldsymbol{Y}) = \eta(\boldsymbol{A}_{m,j})$, all the members of $\boldsymbol{X}$ in that interval are less than all the members of $\boldsymbol{Y}$ in that interval. This will be done in Claim 7. That claim will set us up to show that every map in $\mathcal{I}_j$ which fixes the members of $\boldsymbol{I}_j$ is actually in $\mathcal{I}'_j$.

We now give a few more definitions which will aid in the remaining proofs. Let $q = |I_j|$ and enumerate I_j in increasing order as $\{i_p : p < q\}$. Fix the following notation for the intervals of K_j determined by the members of I_j: Let $I^0 = [0, i_0)$, for each $p < q - 1$ let $I^{p+1} = (i_p, i_{p+1})$, and let $I^q = (i_{q-1}, K_j)$. Thus, K_j is the disjoint union of I_j and the intervals I^p, $p \leq q$. Given $\iota \in \mathcal{I}'_j$ and $(\boldsymbol{X}, \boldsymbol{Y}) = \iota(\boldsymbol{A}_{m,j})$, for $p \leq q$ and $k, l \in I^p$, we say that (k, l) is the *maximal switching pair of ι in I^p* if the following holds:

(a) $l = \max\{i \in I^p : \exists i' \in I^p\ \iota(i') > \iota(K_j + i)\}$ and
(b) $k = \min\{i' \in I^p : \iota(i') > \iota(K_j + l)\}$.

In words, $x^k > y^l$ in $(\boldsymbol{X}, \boldsymbol{Y})$, there are no other members of $(\boldsymbol{X}, \boldsymbol{Y})$ between them, and for every $t > l$ in I^p, y^t is greater than every member of $\boldsymbol{X}$ in the interval I^p. We point out that $x^{\max(I^p)} < y^{\min(I^p)}$ in the structure $(\boldsymbol{X}, \boldsymbol{Y}) = \iota(\boldsymbol{A}_{m,j})$ if and only if there is no maximal switching pair for ι in the interval I^p. This is the configuration we are heading for in Claim 7 below.

For $\iota \in \mathcal{I}'_j$, define the order relation induced by ι, $\rho_\iota : K_j \times K_j \to \{<, =, >\}$, as follows: For $(k, l) \in K_j \times K_j$ and $\rho \in \{<, =, >\}$, define $\boldsymbol{\rho}_\iota(k, l) = \rho$ if and only if $(\iota(k), \iota(K_j + l)) = \rho$.

Claim 6 *(Maximal switching pair can be switched) Let $\tau \in \mathcal{I}'_j$ and $p \leq q$, and let (k, l) be the maximal switching pair in I^p. Then there is a $\sigma \in \mathcal{I}'_j$ such that for all $(s, t) \in K_j \times K_j \setminus \{(k, l)\}$, $\boldsymbol{\rho}_\sigma(s, t) = \boldsymbol{\rho}_\tau(s, t)$, and $\boldsymbol{\rho}_\sigma(k, l) = <$.*

Proof Let $\tau \in \mathcal{I}'_j$, let $p \leq q$, and let (k, l) be the maximal switching pair for τ in I^p. Let $(\boldsymbol{X}, \boldsymbol{Y}) = \tau(\boldsymbol{A}_{m,j})$. Since $l \notin I_j$, Claim 4 implies there is an $\iota \in \mathcal{I}'_j$ such that

$\iota(k) = \iota(K_j + k)$ for all $k \in K_j \backslash \{l\}$, and $\iota(l) < \iota(K_j + l)$. Let $(\boldsymbol{Y}_*, \boldsymbol{Z}) = \iota(\boldsymbol{A}_{m,j})$. Let $e : \boldsymbol{Y} \to (\boldsymbol{X}, \boldsymbol{Y})$ be the identity map on $\boldsymbol{Y}$, and let $f : \boldsymbol{Y}_* \to (\boldsymbol{Y}_*, \boldsymbol{Z})$ be the identity map on $\boldsymbol{Y}_*$.

By the OPFAP, there is a free amalgamation of $(\boldsymbol{X}, \boldsymbol{Y})$ and $(\boldsymbol{Y}_*, \boldsymbol{Z})$ with the order prescribed by the order relation $\boldsymbol{\rho}$, which is now described. Let $\boldsymbol{U}$ denote $(\boldsymbol{X}, \boldsymbol{Y})$ and $\boldsymbol{V}$ denote $(\boldsymbol{Y}_*, \boldsymbol{Z})$. In words, since the only difference between $\boldsymbol{Y}_*$ and $\boldsymbol{Z}$ in $\boldsymbol{V}$ is at their l-th members, we identify $\boldsymbol{Y}_*$ with $\boldsymbol{Y}$ and define $\boldsymbol{\rho}$ between $\boldsymbol{U}$ and $\boldsymbol{Y}_*$ as $\boldsymbol{\rho}_\tau$, and additionally we order $x^k < z^l$ and z^l less than the least member of $\boldsymbol{U}$ above x^k. Precisely, for $(s, t) \in |U| \times |V|$ such that $u^s = y^i$ and $v^t = y^i_*$ for the same $i \in K_j$, define $\boldsymbol{\rho}(s, t) = =$. This induces the relation $\boldsymbol{\rho}(s', t) = \boldsymbol{\rho}_\tau(s', t)$ for all $s' \in |U|$ and $t \in K_j + 1 \backslash \{l + 1\}$. Let s_k be the number in $|U|$ such that $u^{s_k} = x^k$. Note that $|V| = K_j + 1$ and $v^{l+1} = z^l$. Define $\boldsymbol{\rho}(s_k, l+1) = <$, and define $\boldsymbol{\rho}(s_k + 1, l + 1) = >$. The rest of $\boldsymbol{\rho}$ is completely determined by the above relations, since we require $\boldsymbol{\rho}$ to respect the linear orders on $|U|$ and $K_j + 1$. That is, we require that if $\boldsymbol{\rho}(s, t) = <$, then $\boldsymbol{\rho}(s', t') = <$ for all $s' \leq s$ and $t' \geq t$; and if $\boldsymbol{\rho}(s, t) = >$, then $\boldsymbol{\rho}(s', t') = >$ for all $s' \geq s$ and $t' \leq t$.

By the OPFAP, there is a free amalgamation $(g, h, \boldsymbol{W})$ of $(\boldsymbol{Y}, e, \boldsymbol{U}, f, \boldsymbol{V})$ respecting the order $\boldsymbol{\rho}$. There is a copy $\boldsymbol{W}' \cong \boldsymbol{W}$ which is a substructure of $\boldsymbol{C}_{i,j}$ for some i, since each member of $\mathcal{K}_j$ embeds into all but finitely many $\boldsymbol{C}_{i,j}$. Slightly abusing notation, we have that $(g, h, \boldsymbol{W}')$ is a free amalgamation of $(\boldsymbol{Y}, e, \boldsymbol{U}, f, \boldsymbol{V})$ respecting the order $\boldsymbol{\rho}$.

Let $\boldsymbol{U}' = (\boldsymbol{X}', \boldsymbol{Y}')$ denote $g(\boldsymbol{U}) = g(\boldsymbol{X}, \boldsymbol{Y})$ and let $\boldsymbol{V}' = (\boldsymbol{Y}'_*, \boldsymbol{Z}')$ denote $h(\boldsymbol{V}) = h(\boldsymbol{Y}_*, \boldsymbol{Z})$, substructures of $\boldsymbol{W}'$. By choosing C small enough within B, similarly to the proof in Claim 2, we may assume that there are $\boldsymbol{D}_{m,j}, \boldsymbol{E}_{m,j} \in \binom{\boldsymbol{B}_{i',j}}{\boldsymbol{A}_{m,j}}$ such that $\tau(\boldsymbol{D}_{m,j}) = \boldsymbol{U}'$ and $\iota(\boldsymbol{E}_{m,j}) = \boldsymbol{V}'$. Since $\tau \in \mathcal{I}'_j$ and $\boldsymbol{U}' = \tau(\boldsymbol{D}_{m,j})$ it follows that $\boldsymbol{X}' \mathrel{E} \boldsymbol{Y}'$. Likewise, $\iota \in \mathcal{I}'_j$ and $\boldsymbol{V}' = \iota(\boldsymbol{E}_{m,j})$ imply that $\boldsymbol{Y}'_* \mathrel{E} \boldsymbol{Z}'$. Since $\boldsymbol{Y}'$ and $\boldsymbol{Y}'_*$ are the same substructure of $\boldsymbol{W}'$ and E is an equivalence relation, we have $\boldsymbol{X}' \mathrel{E} \boldsymbol{Z}'$.

Let σ be any member of $\mathcal{I}_j$ such that $\sigma(\boldsymbol{A}_{m,j}) \cong (\boldsymbol{X}', \boldsymbol{Z}')$. Again, by choosing C small enough within B, similarly to the proof in Claim 2, we may assume that there is an $\boldsymbol{F}_{m,j} \in \binom{\boldsymbol{B}_{i',j}}{\boldsymbol{A}_{m,j}}$ such that $\sigma(\boldsymbol{F}_{m,j}) = (\boldsymbol{X}', \boldsymbol{Z}')$. Then σ is in $\mathcal{I}'_j$, since $\boldsymbol{X}' \mathrel{E} \boldsymbol{Z}'$. Note that for any such σ, $\boldsymbol{\rho}_\sigma$ is the same as $\boldsymbol{\rho}_\tau$ except at the pair (k, l), where now $\boldsymbol{\rho}_\sigma(k, l) = <$. Thus, there is a $\sigma \in \mathcal{I}'_j$ satisfies the claim. □

Claim 7 *There is an $\eta \in \mathcal{I}'_j$ such that for each $p \leq q$, $\max(\eta''I^p) < \min(\eta''\{K_j + i : i \in I^p\})$, and there are no relations between $\boldsymbol{A}_{m,j} \restriction \eta''(K_j \backslash I_j)$ and $\boldsymbol{A}_{m,j} \restriction \eta''\{K_j + k : k \in K_j \backslash I_j\}$.*

Proof Let $p \leq q$ be given and assume that I^p is nonempty. Let $\tau \in \mathcal{I}'_j$ and (k, l) be the maximal switching pair for τ in the interval I^p. By finitely many applications of Claim 6, there is a $\tau_l \in \mathcal{I}'_j$ such that for all $(s, t) \in K_j \times (K_j \backslash \{l\})$, $\boldsymbol{\rho}_{\tau_l}(s, t) = \boldsymbol{\rho}_\tau(s, t)$; for all $s \leq \max(I^p)$, $\boldsymbol{\rho}_{\tau_l}(s, l) = <$; and for all $s > \max(I^p)$, $\boldsymbol{\rho}_{\tau_l}(s, l) = >$. If $l > \min(I^p)$, then the applications of Claim 6 constructed τ_l so that $\tau_l(k) > \tau_l(l - 1)$. Thus, there is some $k_1 \leq k$ such that $(k_1, l - 1)$ is the maximal switching pair for τ_l in the interval I^p. By finitely many applications of Claim 6, we obtain a $\tau_{l-1} \in \mathcal{I}'_j$ such that for all $(s, t) \in K_j \times (K_j \backslash \{l - 1\})$, $\boldsymbol{\rho}_{\tau_{l-1}}(s, t) = \boldsymbol{\rho}_{\tau_l}(s, t)$; for all $s \leq \max(I^p)$,

$\boldsymbol{\rho}_{\tau_{l-1}}(s, l-1) = <$; and for all $s > \max(I^p)$, $\boldsymbol{\rho}_{\tau_{l-1}}(s, l-1) = >$. Continuing in this manner, we eventually obtain a $\sigma^p \in \mathcal{I}'_j$ such that for all $(s, t) \in K_j \times K_j \backslash I^p \times I^p$, $\boldsymbol{\rho}_{\sigma^p}(s, t) = \boldsymbol{\rho}_\tau(s, t)$; and for all $(s, t) \in I^p \times I^p$, $\boldsymbol{\rho}_{\sigma^p}(s, t) = <$.

By induction on the intervals to build an η as in the claim as follows: Starting with any $\tau \in \mathcal{I}'_j$, by the previous paragraph, there is a $\sigma^0 \in \mathcal{I}'_j$ such that for all $(s, t) \in K_j \times K_j \backslash I^0 \times I^0$, $\boldsymbol{\rho}_{\sigma^0}(s, t) = \boldsymbol{\rho}_\tau(s, t)$; and for all $(s, t) \in I^p \times I^p$, $\boldsymbol{\rho}_{\sigma^0}(s, t) = <$. Given $p < q$ and σ^p, by the previous paragraph, there is a $\sigma^{p+1} \in \mathcal{I}'_j$ such that for all $(s, t) \in K_j \times K_j \backslash I^{p+1} \times I^{p+1}$, $\boldsymbol{\rho}_{\sigma^{p+1}}(s, t) = \boldsymbol{\rho}_{\sigma^p}(s, t)$; and for all $(s, t) \in I^{p+1} \times I^{p+1}$, $\boldsymbol{\rho}_{\sigma^{p+1}}(s, t) = <$.

Let σ denote σ^q. Then $\sigma \in \mathcal{I}'_j$ and for each $p \leq q$, $\max(\sigma'' I^p) < \min(\sigma''\{K_j + i : i \in I^p\})$. However, $\sigma(\boldsymbol{A}_{m,j})$ might not be the free amalgamation of $\boldsymbol{A}_{m,j} \restriction \sigma'' K_j$ and $\boldsymbol{A}_{m,j} \restriction \sigma''[K_j, 2K_j)$ over the structure $\boldsymbol{A}_{m,j} \restriction \sigma'' I_j$. Take structures $(\boldsymbol{X}, \boldsymbol{Y}) \cong (\boldsymbol{X}', \boldsymbol{Z}) \cong \sigma(\boldsymbol{A}_{m,j})$. By the OPFAP, there is a free amalgamation $\boldsymbol{W}$ of $(\boldsymbol{X}, \boldsymbol{Y})$ and $(\boldsymbol{X}', \boldsymbol{Z})$ with the following properties: $\boldsymbol{X}$ and $\boldsymbol{X}'$ are sent to the same substructure, call it $\boldsymbol{X}_*$ of $\boldsymbol{W}$ (hence $\boldsymbol{X} \restriction I_j$ is sent to the same substructure as $\boldsymbol{X}' \restriction I_j$). Moreover, letting $\boldsymbol{Y}_*, \boldsymbol{Z}_*$ denote the copies of $\boldsymbol{Y}$ and $\boldsymbol{Z}$ in $\boldsymbol{W}$, we have that for all $p \leq q$, for all $i, k, l \in I^p$, $x^i_* < y^k_* < z^l_*$. Thus, $\boldsymbol{Y}_* \restriction K_j \backslash I_j$ and $\boldsymbol{Z}_* \restriction K_j \backslash I_j$ have no relations between them in $\boldsymbol{W}$. Thus, the substructure $(\boldsymbol{Y}_*, \boldsymbol{Z}_*)$ in $\boldsymbol{W}$, is the free amalgamation of two copies of $\boldsymbol{A}_{n,j}$ over the substructure $\boldsymbol{A}_{n,j} \restriction I_j$, and for each $p \leq q$, all the members of $\boldsymbol{Y}_*$ in the interval I^p are less than all the members of $\boldsymbol{Z}_*$ in I^p. Since the structure $(\boldsymbol{Y}_*, \boldsymbol{Z}_*)$ is an amalgamation of two copies of $\boldsymbol{A}_{n,j}$, there is an $\eta \in \mathcal{I}_j$ such that $\eta(\boldsymbol{A}_{m,j}) \cong (\boldsymbol{Y}_*, \boldsymbol{Z}_*)$. Since $\sigma \in \mathcal{I}'_j$, we have that $\boldsymbol{X}_* \, E \, \boldsymbol{Y}_*$ and $\boldsymbol{Y}_* \, E \, \boldsymbol{Z}_*$. Thus, $\boldsymbol{Y}_* \, E \, \boldsymbol{Z}_*$. Therefore, η is in $\mathcal{I}'_j$, by Lemma 33. This η satisfies the claim. □

Claim 8 *Let $k \geq m$. Given any $\boldsymbol{X}, \boldsymbol{Y} \in \binom{\boldsymbol{C}_{k,j}}{\boldsymbol{A}_{n,j}}$ such that for all $i \in I_n$, $x^i = y^i$, there is a $\tau \in \mathcal{I}'_j$ such that for some $\boldsymbol{W} \in \binom{\boldsymbol{B}_{k',j}}{\boldsymbol{A}_{m,j}}$, $\tau(\boldsymbol{W}) = (\boldsymbol{X}, \boldsymbol{Y})$.*

Proof Let $\boldsymbol{X}, \boldsymbol{Y} \in \binom{\boldsymbol{C}_{k,j}}{\boldsymbol{A}_{n,j}}$ be such that for all $i \in I_j$, $x^i = y^i$. Take $\boldsymbol{Z} \in \binom{\boldsymbol{B}_{k',j}}{\boldsymbol{A}_{n,j}}$ such that for each $i \in I_j$, $z^i = x^i$, and for each $p \leq q$, $\max \boldsymbol{Z} \restriction I^p < \min(\boldsymbol{X} \restriction I^p, \boldsymbol{Y} \restriction I^p)$, and any relations between $\boldsymbol{Z}$ and $\boldsymbol{X}$ and any relations between $\boldsymbol{Z}$ and $\boldsymbol{Y}$ involve only members of their universes with indices in I_j. (By the OPFAP and possibly thinning C again below B, such a $\boldsymbol{Z}$ exists.) Let $\boldsymbol{V}$ be the substructure of $\boldsymbol{B}_{k',j}$ determined by the universe $|\boldsymbol{X}| \cup |\boldsymbol{Z}|$; and let $\boldsymbol{W}$ be the substructure of $\boldsymbol{B}_{k',j}$ determined by the universe $|\boldsymbol{Y}| \cup |\boldsymbol{Z}|$. Let η be the member of $\mathcal{I}'_j$ from Claim 7. Then both $\boldsymbol{V}$ and $\boldsymbol{W}$ are isomorphic to the structure $\eta(\boldsymbol{A}_{m,j})$. Since $\eta \in \mathcal{I}'_j$, we have $\boldsymbol{Z} \, E \, \boldsymbol{X}$ and $\boldsymbol{Z} \, E \, \boldsymbol{Y}$. Since E is an equivalence relation, it follows that $\boldsymbol{X} \, E \, \boldsymbol{Y}$. It follows that any $\tau \in \mathcal{I}_j$ for which there is a $\boldsymbol{D} \in \binom{\boldsymbol{B}_{k',j}}{\boldsymbol{A}_{m,j}}$ such that $\tau(\boldsymbol{D}) = (\boldsymbol{X}, \boldsymbol{Y})$ is in $\mathcal{I}'_j$. Therefore, each $\tau \in \mathcal{I}_j$ which is fixed on indices in I_j is also in $\mathcal{I}'_j$. □

By Claim 8, the following is immediate.

Claim 9 *For each $k \geq n$, for all $X(n), Y(n) \in \mathcal{R}(n)|C(k)$, we have $\boldsymbol{X}_{n,j} \, E \, \boldsymbol{Y}_{n,j}$ if and only $\boldsymbol{X}_{n,j} \, E_{I_j} \, \boldsymbol{Y}_{n,j}$.*

Proof Let $k \geq n$ and $X(n), Y(n) \in \mathcal{R}(n)|C(k)$. If $\boldsymbol{X}_{n,j}\ E\ \boldsymbol{Y}_{n,j}$, then there is an $\iota \in \mathcal{I}'_j$ and a $\boldsymbol{U} \in \binom{\boldsymbol{B}_{k',j}}{\boldsymbol{A}_{m,j}}$ such that $(\boldsymbol{X}_{n,j}, \boldsymbol{Y}_{n,j}) = \iota(\boldsymbol{U})$, by Lemma 33. Since $\iota \in \mathcal{I}'_j$, we have that for each $i \in I_j$, $\iota(i) = \iota(K_{n,j}+i)$; hence, for each $i \in I_j$, $x^i_{n,j} = y^i_{n,j}$. Therefore, $\boldsymbol{X}_{n,j}\ E_{I_j}\ \boldsymbol{Y}_{n,j}$. Conversely, suppose $\boldsymbol{X}_{n,j}\ E_{I_j}\ \boldsymbol{Y}_{n,j}$. Let $\iota \in \mathcal{I}_j$ and $\boldsymbol{W}_{m,j} \in \binom{\boldsymbol{B}_{k',j}}{\boldsymbol{A}_{m,j}}$ be such that $\iota(\boldsymbol{W}_{m,j}) = (\boldsymbol{X}_{n,j}, \boldsymbol{Y}_{n,j})$. By Claim 8, $\iota \in \mathcal{I}'_j$. Thus, $\boldsymbol{X}_{n,j}\ E\ \boldsymbol{Y}_{n,j}$. □

It follows from Claim 9 and Lemma 33 that for each $\tau \in \mathcal{I}_j$, τ is in $\mathcal{I}'_j$ if and only if for all $i \in I_j$, $\tau(i) = \tau(K_j+1)$. This completes the Base Case.

Induction Hypothesis. Given n such that $J_n \geq M$, $N \leq M$, and $L \in [J_n]^N$, letting E be an equivalence relation such that $E \subseteq \bigcup_{k\geq n} \binom{(\boldsymbol{A}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}} \times \binom{(\boldsymbol{A}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}}$, the following hold. Fix any m large enough that for each $j \in L$, all amalgamations of two copies of $\boldsymbol{A}_{n,j}$ can be embedded into $\boldsymbol{A}_{m,j}$, and let $C \leq B$ and $\mathcal{I}' \subseteq \mathcal{I}$ be obtained as in Lemma 33 and similarly as in Claims 1 and 2. Letting, for $j \in L$,

$$I_j = \{i \in K_j : \forall \iota_j \in \mathcal{I}'_j, \iota_j(i) = \iota_j(K_j+i)\}, \tag{7}$$

the following hold:

(a) $\mathcal{I}' = \Pi_{j\in L}\mathcal{I}'_j$, where for each $j \in L$, $\mathcal{I}'_j = \{\iota_j : \iota \in \mathcal{I}'\}$.
(b) When restricted below C, $E = E_{(I_j)_{j\in L}}$.

Induction Step. $M+1$. Let $n < \omega$ be such that $M+1 \geq J_n$. Let $L \in [J_n]^{M+1}$, and let E be an equivalence relation such that $E \subseteq \bigcup_{k\geq n} \binom{(\boldsymbol{A}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}} \times \binom{(\boldsymbol{A}_{k,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}}$. Let m be large enough that for each $j \in L$, all amalgamations of two copies of $\boldsymbol{A}_{n,j}$ can be embedded into $\boldsymbol{A}_{m,j}$. Let $l = \max(L)$, and let $L' = L\setminus\{l\}$. We start by fixing $B, C \in \mathcal{R}$, with $C \leq B$, and $\mathcal{I}'$ satisfying Lemma 33. For each $j \in L$, let $\mathcal{I}'_j$ denote the collection of those $\iota_j \in \mathcal{I}_j$ for which there exists a $\tau = (\tau_k : k \in L) \in \mathcal{I}'$ such that $\iota_j = \tau_j$.

For each $W \in \mathcal{R}$ and each $K_l \subseteq M_l := \|\boldsymbol{A}_{m,l}\|$ such that $\boldsymbol{A}_{m,l} \restriction K_l \cong \boldsymbol{A}_{n,l}$, by the induction hypothesis there is a $V \leq W$ such that for each $X \leq V$, E restricted to the copies of $(\boldsymbol{A}_{n,j})_{j\in L}$ in $(\boldsymbol{X}_{m,j})_{j\in L'}{}^\frown(\boldsymbol{X}_{m,l} \restriction K_l)$ is canonical. By the OPFAP and the definition of generating sequence, for any $K_l, K'_l \subseteq M_l$ satisfying $\boldsymbol{A}_{m,l} \restriction K_l \cong \boldsymbol{A}_{m,l} \restriction K'_l \cong \boldsymbol{A}_{n,l}$, there is a p large enough so that there are structures $\boldsymbol{Y}_{m,l}, \boldsymbol{Z}_{m,l}$ in $\binom{\boldsymbol{A}_{p,l}}{\boldsymbol{A}_{m,l}}$ with $\boldsymbol{Y}_{m,l} \restriction K_l = \boldsymbol{Z}_{m,l} \restriction K'_l$. Thus, possibly thinning B, we have that the canonical equivalence relation on $\binom{(\boldsymbol{B}_{p,j})_{j\in L'}{}^\frown(\boldsymbol{D}_{n,l})}{(\boldsymbol{A}_{n,j})_{j\in L}}$ is the same for all $p \geq m$ and each fixed $\boldsymbol{D}_{n,l} \in \binom{\boldsymbol{B}_{p,l}}{\boldsymbol{A}_{n,l}}$.

Let $\mathcal{T}_{L'}$ denote the collection of $\tau = (\tau_j : j \in L')$ (where each $\tau_j \in \mathcal{I}_j$) which give the canonical equivalence relation on $\binom{(\boldsymbol{B}_{p,j})_{j\in L'}{}^\frown(\boldsymbol{C}_{n,l})}{(\boldsymbol{A}_{n,j})_{j\in L}}$, for each $p \geq m$ and $\boldsymbol{C}_{n,l} \in \binom{\boldsymbol{B}_{p,l}}{\boldsymbol{A}_{n,l}}$. By (a) of the induction hypothesis, $\mathcal{T}_{L'} = \Pi_{j\in L'}\mathcal{T}_j$, where for each $j \in L'$, $\mathcal{T}_j = \{\tau_j : \tau \in \mathcal{T}_{L'}\}$. For each $j \in L'$, let $H_j = \{i \in K_j : \forall \tau \in \mathcal{T}_{L'},\ \tau_j(i) = \tau_j(K_j+i)\}$. By (b) of the induction hypothesis, below B the canonical equivalence relation when the l-th coordinate is fixed is $E_{(H_j)_{j\in L'}}$.

Likewise, for each $W \in \mathcal{R}$ and each collection $K_j \subseteq M_j := \|\boldsymbol{A}_{m,j}\|$ $(j \in L')$ such that $\boldsymbol{A}_{m,j} \restriction K_j \cong \boldsymbol{A}_{n,j}$, by the induction hypothesis, there is a $V \leq W$ such that for each $X \leq V$, E restricted to the copies of $(\boldsymbol{A}_{n,j})_{j\in L}$ in $(\boldsymbol{X}_{m,j} \restriction K_j)_{j\in L'}{}^\frown(\boldsymbol{X}_{m,l})$ is canonical. By the OPFAP and the definition of a generating sequence, it follows that for each $W \in \mathcal{R}$, there is a $V \leq W$ such that for all $j \in L'$, whenever $K_j, K_j' \subseteq M_j$ satisfy $\boldsymbol{A}_{m,j} \restriction K_j \cong \boldsymbol{A}_{m,j} \restriction K_j' \cong \boldsymbol{A}_{n,j}$, then there are $\boldsymbol{Y}_{m,j}, \boldsymbol{Z}_{m,j} \in \binom{\boldsymbol{W}_{p,j}}{\boldsymbol{A}_{n,j}}$, for some $p \geq m$, such that $\boldsymbol{Y}_{m,j} \restriction K_j = \boldsymbol{Z}_{m,j} \restriction K_j'$. Thus, possibly thinning B, we may assume that the equivalence relation is the same canonical one on each set $\binom{(\boldsymbol{D}_{n,j})_{j\in L'}{}^\frown(\boldsymbol{B}_{p,l})}{(\boldsymbol{A}_{n,j})_{j\in L}}$ for all $p \geq m$ and each fixed $(\boldsymbol{D}_{n,j})_{j\in L'} \in \binom{(\boldsymbol{B}_{p,j})_{j\in L'}}{(\boldsymbol{A}_{n,j})_{j\in L'}}$.

Let $\mathcal{T}_l$ denote the collection of $\tau_l \in \mathcal{I}_l$ which give the canonical equivalence relation on the copies of $(\boldsymbol{A}_{n,j})_{j\in L}$ in $\binom{(\boldsymbol{D}_{n,j})_{j\in L'}{}^\frown(\boldsymbol{B}_{p,l})}{(\boldsymbol{A}_{n,j})_{j\in L}}$ for all $p \geq m$ and each fixed $(\boldsymbol{D}_{n,j})_{j\in L'} \in \binom{(\boldsymbol{B}_{p,j})_{j\in L'}}{(\boldsymbol{A}_{n,j})_{j\in L'}}$. Let $H_l = \{i \in K_l : \forall \tau_l \in \mathcal{T}_l,\ \tau_l(i) = \tau_l(K_l + i)\}$. Thus, below B, the canonical equivalence relation when the l-th coordinate is fixed is E_{H_l}.

By the induction hypothesis, $\mathcal{T}_{L'} = \prod_{j\in L'} \mathcal{T}_j$. Thus, $\mathcal{T}_{L'} \times \mathcal{T}_l = \prod_{j\in L} \mathcal{T}_j$. Moreover, each I_j must be contained in H_j, for each $j \in L$.

Claim 10 $\prod_{j\in L} \mathcal{T}_j \subseteq \mathcal{I}'$. *Hence, below* C, $E_{(H_j)_{j\in L}} \subseteq E$.

Proof Given any $\tau = (\tau_j : j \in L) \in \prod_{j\in L} \mathcal{T}_j$ and $((\boldsymbol{X}_{n,j})_{j\in L}, (\boldsymbol{Y}_{n,j})_{j\in L}) = \tau((\boldsymbol{C}_{m,j})_{j\in L})$, we see that $(\boldsymbol{X}_{n,j})_{j\in L}\ E\ (\boldsymbol{X}_{n,j})_{j\in L'}{}^\frown\boldsymbol{Y}_{n,l}\ E\ (\boldsymbol{Y}_{n,j})_{j\in L}$. Thus, $\tau \in \mathcal{I}'$.

Suppose that $(\boldsymbol{X}_{n,j})_{j\in L}, (\boldsymbol{Y}_{n,j})_{j\in L} \in \binom{(\boldsymbol{C}_{p,j})_{j\in L}}{(\boldsymbol{A}_{n,j})_{j\in L}}$ satisfy $(\boldsymbol{X}_{n,j})_{j\in L}\ E_{(H_j)_{j\in L}}\ (\boldsymbol{Y}_{n,j})_{j\in L}$, where $p \geq n$. Let $\boldsymbol{Z}_{n,j} = \boldsymbol{X}_{n,j}$ for each $j \in L'$, and let $\boldsymbol{Z}_{n,l} = \boldsymbol{Y}_{n,l}$. Then $(\boldsymbol{X}_{n,j})_{j\in L}\ E\ (\boldsymbol{Z}_{n,j})_{j\in L}$, and $(\boldsymbol{Z}_{n,j})_{j\in L}\ E\ (\boldsymbol{Y}_{n,j})_{j\in L}$. Thus, $(\boldsymbol{X}_{n,j})_{j\in L}\ E\ (\boldsymbol{Y}_{n,j})_{j\in L}$, by transitivity of E. □

Claim 11 *Below* C, $\mathcal{I}' \subseteq \prod_{j\in L} \mathcal{T}_j$.

Proof Let $\iota := (\iota_j : j \in L) \in \mathcal{I}'$ and let $((\boldsymbol{X}_{n,j})_{j\in L}, (\boldsymbol{Y}_{n,j})_{j\in L}) = \iota((\boldsymbol{C}_{m,j})_{j\in L})$. Then $(\boldsymbol{X}_{n,j})_{j\in L}\ E\ (\boldsymbol{Y}_{n,j})_{j\in L}$. Fixing $((\boldsymbol{X}_{n,j})_{j\in L'}, (\boldsymbol{Y}_{n,j})_{j\in L'})$ and running the arguments for the Base Case on coordinate l, we obtain Claim 7 on $\boldsymbol{C}_{m,l}$. Take $\eta_l \in \mathcal{T}_l$ as in Claim 7. Take a $\boldsymbol{Z}_{n,l} \in \binom{\boldsymbol{B}_{m',l}}{\boldsymbol{A}_{n,l}}$ such that both $(\boldsymbol{X}_{n,l}, \boldsymbol{Z}_{n,l}) = \eta_l(\boldsymbol{V}_{m,l})$ and $(\boldsymbol{Y}_{n,l}, \boldsymbol{Z}_{n,l}) = \eta(\boldsymbol{W}_{m,l})$ for some $\boldsymbol{V}_{m,l}, \boldsymbol{W}_{m,l} \in \binom{\boldsymbol{B}_{m',l}}{\boldsymbol{A}_{n,l}}$. In particular, $\boldsymbol{X}_{n,l} \restriction H_l = \boldsymbol{Z}_{n,l} \restriction H_l = \boldsymbol{Y}_{n,l} \restriction H_l$. It follows that $(\boldsymbol{X}_{n,j})_{j\in L'}\ E\ (\boldsymbol{X}_{n,j})_{j\in L'}{}^\frown\boldsymbol{Z}_{n,l}$, and $(\boldsymbol{Y}_{n,j})_{j\in L'}\ E\ (\boldsymbol{Y}_{n,j})_{j\in L'}{}^\frown\boldsymbol{Z}_{n,l}$. Therefore, $(\boldsymbol{X}_{n,j})_{j\in L'}{}^\frown\boldsymbol{Z}_{n,l}\ E\ (\boldsymbol{Y}_{n,j})_{j\in L'}{}^\frown\boldsymbol{Z}_{n,l}$, which implies that $(\iota_j : j \in L') \in \mathcal{T}_{L'}$. Hence, for each $j \in L'$, ι_j is in $\mathcal{T}_j$.

By a similar argument, say fixing $((\boldsymbol{X}_{n,j})_{j\in L\setminus\{0\}}, (\boldsymbol{Y}_{n,j})_{j\in L\setminus\{0\}})$, we find that ι_l is in $\mathcal{T}_l$. Therefore, $\iota \in \prod_{j\in L} \mathcal{T}_j$. □

Therefore, $\mathcal{I}' = \prod_{j\in L} \mathcal{T}_j$. Hence, below C, E is given by $E_{(H_j)_{j\in L}}$. We conclude by showing that the induction hypotheses (a) and (b) are satisfied for this stage.

If ι_j is in $\mathcal{I}'$, then there was some $\tau = (\tau_l : l \in L) \in \mathcal{I}'$ such that $\iota_j = \tau_j$. Since $\mathcal{I}' = \prod_{j\in L} \mathcal{T}_j$, ι_j must be in $\mathcal{T}_j$. Conversely, if $\tau_j \in \mathcal{T}_j$, then taking any $\tau_l \in \mathcal{T}_l$ for $l \in L\setminus\{j\}$, we have that $(\tau_l : l \in L)$ is in $\mathcal{I}'$. Thus, τ_j is in $\mathcal{I}_j'$. Therefore, $\mathcal{I}' = \prod_{l\in L} \mathcal{I}_l'$, so (a) holds.

Define I_j to be the set of all $i \in K_j$ such that for all $\iota_j \in \mathcal{I}'_j$, $\iota_j(i) = \iota_j(K_j + i)$. It was shown above that each $I_j \subseteq H_j$. Suppose there is an $i \in H_j \setminus I_j$. There is an $(\iota_l : l \in L) \in \mathcal{I}'$ such that $\iota_j(i) \neq K_j + i$. Every $(\tau_l : l \in L) \in \prod_{l \in L} \mathcal{T}_l$ must have $\tau_j(i) = \tau_j(K_j + i)$. But $\prod_{l \in L} \mathcal{T}_l$ equals $\mathcal{I}'$, a contradiction. Therefore, each $I_j = H_j$, hence (b) holds.

This finishes the proof of the theorem. □

Remark Sokič has pointed out that it seems to be sufficient to assume the weaker Order-preserving Strong Amalgamation Property, the same definition as OPFAP except only requiring the amalgamations to be strong, not necessarily free.

6 General Ramsey-classification theorem for topological Ramsey spaces constructed from generating sequences

We prove a general Ramsey-classification theorem, Theorem 38, for equivalence relations on fronts for the class of the topolgical Ramsey spaces introduced in Sect. 3, where the Fraïssé classes have the OPFAP. Theorem 38 extends Theorem 4.14 from [11] for canonical equivalence relations on the space $\mathcal{R}_1$ to the more general class of topological Ramsey spaces constructed from a generating sequence. As the proof here closely follows that in [11], we shall omit those proofs which follow by straightforward modifications of arguments in that paper. The essential new ingredient here is that the building blocks for Theorem 38 are the canonical equivalence relations from Theorem 31, and handling this shall require some care.

Throughout this section, let $1 \leq J \leq \omega$, and $\mathcal{K}_j$, $j \in J$, be Fraïssé classes of finite ordered relational structures with the Ramsey property and the Order-Prescribed Free Amalgamation Property. Let $\langle \boldsymbol{A}_k : k < \omega \rangle$ be a fixed generating sequence, and let $\mathcal{R}$ denote the topological Ramsey space $\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega \rangle)$. Recall that for $j \in J_k$, $K_{k,j}$ denotes the cardinality of the structure $\boldsymbol{A}_{k,j}$, and for any structure $\boldsymbol{B}_{k,j} \cong \boldsymbol{A}_{k,j}$, we let $\{b^i_{k,j} : i < K_{k,j}\}$ denote the enumeration of the universe of $\boldsymbol{B}_{k,j}$ in increasing order.

Definition 34 (Canonical projection maps on blocks) *Let $k < \omega$ be given. For $\boldsymbol{B}_{k,j} \in \binom{\boldsymbol{A}_{n,j}}{\boldsymbol{A}_{k,j}}$ and $I \subseteq K_{k,j}$, let $\pi_I(\boldsymbol{B}_{k,j}) = \boldsymbol{B}_{k,j} \restriction \{b^i_{k,j} : i \in I\}$, the substructure of $\boldsymbol{B}_{k,j}$ with universe $\{b^i_{k,j} : i \in I\}$.*

For $B(k) = \langle n, (\boldsymbol{B}_{k,j})_{j \in J_k} \rangle \in \mathcal{R}(k)$, we define the following projection maps. Given $I_{k,j} \subseteq K_{k,j}$, $j \in J_k$, let

$$\pi_{(I_{k,j})_{j \in J_k}}(B(k)) = \langle n, (\pi_{I_{k,j}}(\boldsymbol{B}_{k,j}))_{j \in J_k} \rangle, \tag{8}$$

and let

$$\pi_{<>}(B(k)) = \langle \rangle, \tag{9}$$

where $\langle \rangle$ denotes the empty sequence.

We slightly abuse notation by associating $\langle n, (\emptyset)_{j\in J_k}\rangle$ *with* $\langle n\rangle$*. We define the depth projection map as*

$$\pi_{\text{depth}}(B(k)) = \langle n\rangle, \tag{10}$$

the depth of $B(k)$ *in* $\mathbb{A}$*. Then when* $I_{k,j} = \emptyset$ *for all* $j \in J_k$*, we associate* $\pi_{(I_{k,j})_{j\in J_k}}(B(k))$ *with* $\pi_{\text{depth}}(B(k))$*. Let*

$$\Pi(k) = \{\pi_{<>}\} \cup \{\pi_{(I_{k,j})_{j\in J_k}} : \forall j \in J_k,\ I_{k,j} \subseteq K_{k,j}\}. \tag{11}$$

The canonical equivalence relations on blocks are induced by the canonical projection maps as follows.

Definition 35 (Canonical equivalence relations on blocks) *Let* $k < \omega$*, and* $B(k), C(k) \in \mathcal{R}(k)$*. For* $I_{k,j} \subseteq K_{k,j}$*,* $j \in J_k$*, define*

$$B(k)\ E_{(I_{k,j})_{j\in J_k}}\ C(k) \longleftrightarrow \pi_{(I_{k,j})_{j\in J_k}}(B(k)) = \pi_{(I_{k,j})_{j\in J_k}}(C(k)). \tag{12}$$

Define

$$B(k)\ E_{<>}\ C(k) \longleftrightarrow \pi_{<>}(B(k)) = \pi_{<>}(C(k)). \tag{13}$$

Thus, $E_{<>} = \mathcal{R}(k) \times \mathcal{R}(k)$*. We also define*

$$B(k)\ E_{\text{depth}}\ C(k) \longleftrightarrow \pi_{\text{depth}}(B(k)) = \pi_{\text{depth}}(C(k)). \tag{14}$$

When $I_{k,j} = \emptyset$ *for all* $j \in J_k$*, then* E_{depth} *is a simplified notation for* $E_{(I_{k,j})_{j\in J_k}}$*, as in this case, they are the same equivalence relation.*

The collection of canonical equivalence relations *on* $\mathcal{R}(k)$ *is*

$$\mathcal{E}(k) = \{E_{<>}\} \cup \{E_{(I_{k,j})_{j\in J_k}} : \forall j \in J_k,\ I_{k,j} \subseteq K_{k,j}\}. \tag{15}$$

For the following definitions, let $X \in \mathcal{R}$, $\mathcal{F}$ be a front on $[\emptyset, X]$, and φ be a function on $\mathcal{F}$.

Definition 36 *We shall say that* φ *is* inner *iff for each* $b \in \mathcal{F}$*,* $\varphi(b) = \bigcup_{i<|b|} \pi_{r_i(b)}(b(i))$*, where each* $\pi_{r_i(b)}$ *is some member of* $\Pi(i)$*.*

Thus, for $b = \langle\langle n_0, (\boldsymbol{B}_{0,j})_{j\in J_0}\rangle, \dots, \langle n_{k-1}, (\boldsymbol{B}_{k-1,j})_{j\in J_{k-1}}\rangle\rangle$, $\varphi(b) = \{\langle\rangle\} \cup \{\langle n_l, (\boldsymbol{C}_{l,j})_{j\in J_l}\rangle : l \in L\}$, for some subset $L \subseteq k$, and some (possibly empty) substructures $\boldsymbol{C}_{l,j}$ of $\boldsymbol{B}_{l,j}$. That is, φ is inner if it picks out a subsequence of substructures from a given b.

For $l < |b|$, let $\varphi(b) \restriction r_l(b)$ denote $\bigcup_{i<l} \pi_{r_i(b)}(b(i))$, the *initial segment* of $\varphi(b)$ which is obtained from $r_l(b)$. For $b, c \in \mathcal{F}$, we shall say that $\varphi(c)$ is a *proper initial segment* of $\varphi(b)$, and write $\varphi(c) \sqsubset \varphi(b)$, if there is an $l < |b|$ such that $\varphi(c) = \varphi(b) \restriction r_l(b) \neq \varphi(b)$.

Definition 37 *An inner map φ is* Nash-Williams *if whenever $b, c \in \mathcal{F}$ and $\varphi(b) \neq \varphi(c)$, then $\varphi(c) \not\sqsubset \varphi(b)$.*

An equivalence relation R on $\mathcal{F}$ is canonical *if there is an inner, Nash-Williams map φ on $\mathcal{F}$ such that for all $b, c \in \mathcal{F}$, $b \, E \, c \Longleftrightarrow \varphi(b) = \varphi(c)$.*

Remark As for the topological Ramsey spaces considered in [11] and [12], here too there can be different inner, Nash-Williams maps which represent the same canonical equivalence relation. However, there will be one maximal such map, maximality being with respect to the embedding relation on the Fraïssé classes. (See Remark 4.23 and Example 4.24 in [11].) As the maximal such map is the one useful for classifying the initial Tukey structures below an ultrafilter associated with a Ramsey space, we consider the maximal inner Nash-Williams map to be the canonizing map.

We fix the following notation, useful in this and subsequent sections.

Notation For $X \in \mathcal{R}$ and $b, s, t \in \mathcal{AR}$, fix the following notation. If $s \sqsubseteq b$, write $b \backslash s \leq_{\text{fin}} X$ if the blocks in b not in s all come from blocks of X; precisely, if $b \backslash s = \langle \langle n_l, (\boldsymbol{B}_{l,j})_{j \in J_l} \rangle : k \leq l < m \rangle$, then for each $k \leq l < m$, there is a block $\langle n_l, (\boldsymbol{X}_{p,j})_{j \in J_p} \rangle = X(p)$, for some p, such that $(\boldsymbol{B}_{l,j})_{j \in J_l} \leq (\boldsymbol{X}_{p,j})_{j \in J_l}$. For a set $\mathcal{F} \subseteq \mathcal{AR}$, define the following. Let $\mathcal{F}_s = \{b \in \mathcal{F} : s \sqsubseteq b\}$, $\mathcal{F}_s | X = \{b \in \mathcal{F}_s : b \backslash s \leq_{\text{fin}} X\}$.

For $b(k) = \langle n_k, (\boldsymbol{B}_{k,j})_{j \in J_k} \rangle \in \mathcal{R}(k)$, we let $\text{depth}_{\mathbb{A}}(b(k)) = n_k$, the depth of the block $b(k)$ in $\mathbb{A}$. For $\mathcal{F} \subseteq \mathcal{AR}$, define $\mathcal{F}_s | X/t = \{b \in \mathcal{F}_s | X : \text{depth}_{\mathbb{A}}(b(|s|)) > \text{depth}_{\mathbb{A}}(t)\}$. Similarly, let $r_{|s|+1}[s, X]/t = \{b \in r_{|s|+1}[s, X] : \text{depth}_{\mathbb{A}}(b(|s|)) > \text{depth}_{\mathbb{A}}(t)\}$.

It is important to note that b being in $\mathcal{F}_s | X$ or $\mathcal{F}_s | X/t$ does not imply that $b \leq_{\text{fin}} X$; it only means that the blocks of b above s comes from within X.

The next theorem is the main result of this section.

Theorem 38 *Let $1 \leq J \leq \omega$, and $\mathcal{K}_j$, $j \in J$, be Fraïssé classes of finite ordered relational structures with the Ramsey property and the Order-Prescribed Free Amalgamation Property. Let $\langle \boldsymbol{A}_k : k < \omega \rangle$ be a fixed generating sequence, and let $\mathcal{R}$ denote the topological Ramsey space $\mathcal{R}(\langle \boldsymbol{A}_k : k < \omega \rangle)$.*

Suppose $A \in \mathcal{R}$, $\mathcal{F}$ is a front on $[\emptyset, A]$, and R is an equivalence relation on $\mathcal{F}$. Then there exists $C \leq A$ such that R is canonical on $\mathcal{F}|C$.

Proof Let $f : \mathcal{F} \to \mathbb{N}$ be any map which induces R. We begin by reviewing the concepts of mixing and separating, first introduced in [22] and used in a more general form in [11,12]. Let $\hat{\mathcal{F}}$ denote $\{r_n(b) : b \in \mathcal{F}, n \leq |b|\}$, the collection of all initial segments of members of $\mathcal{F}$. For $s, t \in \hat{\mathcal{F}}$, we shall say that X *separates s and t* if and only for all $b \in \mathcal{F}_s | X/t$ and $c \in \mathcal{F}_t | X/s$, $f(b) \neq f(c)$. X *mixes s and t* if and only if there is no $Y \leq X$ which separates s and t. X *decides for s and t* if and only if either X separates s and t, or else X mixes s and t.

The proofs of the following Lemmas 39–41 are omitted, as they are the same as the proofs of the analogous statements in [11].

Lemma 39 *(Transitivity of mixing) For every $X \in \mathcal{R}$ and every $s, t, u \in \hat{\mathcal{F}}$, if X mixes s and t and X mixes t and u, then X mixes s and u.*

Since mixing is trivially reflexive and symmetric, it is an equivalence relation. We shall say that a property $P(s, X)$ ($s \in \mathcal{AR}$, $X \in \mathcal{R}$) is *hereditary* if whenever $P(s, X)$ holds, then $P(s, Y)$ holds for all $Y \leq X$. Likewise, $P(s, t, X)$ is *hereditary* if whenever $P(s, t, X)$ holds, then $P(s, t, Y)$ holds for all $Y \leq X$.

Lemma 40 *(Diagonalization for Hereditary Properties)*

(1) Suppose $P(\cdot, \cdot)$ is a hereditary property, and that for every $X \in \mathcal{R}$ and every $s \in \mathcal{AR}|X$, there exists $Y \leq X$ such that $P(s, Y)$. Then for every $X \in \mathcal{R}$ there exists $Y \leq X$ such that $P(s, Z)$ holds, for every $s \in \mathcal{AR}|Y$ and every $Z \leq Y$.

(2) Suppose $P(\cdot, \cdot, \cdot)$ is a hereditary property, and that for every $X \in \mathcal{R}$ and all $s, t \in \mathcal{AR}|X$, there exists $Y \leq X$ such that $P(s, t, Y)$ holds. Then, for every $X \in \mathcal{R}$ there exists $Y \leq X$ such that $P(s, t, Z)$ holds, for all $s, t \in \mathcal{AR}|Y$ and every $Z \leq Y$.

Lemma 41 *For each $A \in \mathcal{R}$ there is a $B \leq A$ such that B decides for all $s, t \in \hat{\mathcal{F}}|B$.*

Possibly shrinking A, we may assume that $A \in \mathcal{R}$ satisfies Lemma 41. We now introduce some notation useful for arguments applying the Nash-Williams Theorem.

Notation For $i \leq k < \omega$, we define the projection map $\pi_{\boldsymbol{A}_i} : \mathcal{R}(k) \to \mathcal{R}(i)$ as follows. For $X(k) = \langle n_k, (\boldsymbol{X}_{k,j})_{j \in J_k} \rangle \in \mathcal{R}(k)$, let $\pi_{\boldsymbol{A}_i}(X(k))$ denote $\langle n_k, (\boldsymbol{Y}_{i,j})_{j \in J_i} \rangle$, where for each $j \in J_i$, $\boldsymbol{Y}_{i,j}$ is the projection of $\boldsymbol{X}_{k,j}$ to the lexicographic leftmost copy of $\boldsymbol{A}_{i,j}$ within $\boldsymbol{X}_{k,j}$.

Claim 12 *For each $s \in (\hat{\mathcal{F}} \setminus \mathcal{F})|A$ and each $X \leq A$, there is a $Z \leq X$ and an equivalence relation $E_s \in \mathcal{E}(|s|)$ such that the following holds: Whenever $x, y \in \mathcal{R}(|s|)|Z/s$, letting $a = s^\frown x$ and $b = s^\frown y$, we have that Z mixes a and b if and only if $x \; E_s \; y$.*

Proof Let $n = |s|$ and $X \leq A$ be given. Let R_s be the following relation on $\mathcal{R}(n)|A/s$. For all $x, y \in \mathcal{R}(n)|A/s$,

$$x \; R_s \; y \iff A \text{ mixes } s^\frown x \text{ and } s^\frown y. \tag{16}$$

Define $\mathcal{X} = \{Y \leq X : A \text{ mixes } s^\frown Y(n) \text{ and } s^\frown \pi_{\boldsymbol{A}_n}(Y(n+1))\}$. By the Abstract Nash-Williams Theorem, there is a $Y \leq X$ such that $[\emptyset, Y] \subseteq \mathcal{X}$ or $[\emptyset, Y] \cap \mathcal{X} = \emptyset$.

Suppose $[\emptyset, Y] \subseteq \mathcal{X}$. Then for all $x, y \in \mathcal{R}(n)|Y/s$, we have $x \; R_s \; y$. Fix $x, y \in \mathcal{R}(n)|Y/s$, let $a = s^\frown x$, $b = s^\frown y$, and take $Z_1, Z_2 \leq Y$ such that $Z_1(n) = x$, $Z_2(n) = y$, and $Z_1(n+1) = Z_2(n+1)$. Then $x \; R_s \; y$ follows from the fact that $Z_1, Z_2 \in \mathcal{X}$ and by transitivity of mixing. In this case the proof of the claim finishes by taking $Z = Y$ and $E_s = E_{<>}$.

Suppose now that $[\emptyset, Y] \cap \mathcal{X} = \emptyset$. Then for all $x, y \in \mathcal{R}(n)|Y/s$, we have $x \; R_s \; y \to \text{depth}_{\mathbb{A}}(x) = \text{depth}_{\mathbb{A}}(y)$. Let m be large enough that all possible configurations of isomorphic copies of $(\boldsymbol{A}_{n,j})_{j \in J_n}$ can be embedded into $(\boldsymbol{A}_{m,j})_{j \in J_n}$. Let $\mathcal{I}_n$ denote the collection of all sequences $(I_{n,j})_{j \in J_n}$, where each $I_{n,j} \subseteq K_{n,j}$. (Recall that $K_{n,j}$ is the cardinality of the structure $\boldsymbol{A}_{n,j}$ from the fixed generating sequence.) For each $\mathbb{I} \in \mathcal{I}_n$, define

$$\mathcal{Y}_{\mathbb{I}} = \{Z \le Y : \forall x, y \in \mathcal{R}(n)|Z(m)/s \ (A \text{ mixes } s^\frown x \text{ and } s^\frown y \leftrightarrow \pi_{\mathbb{I}}(x) = \pi_{\mathbb{I}}(y))\} \tag{17}$$

Let $\mathcal{Y} = [\emptyset, Y] \setminus \bigcup_{\mathbb{I}} \mathcal{Y}_{\mathbb{I}}$. Notice that $\mathcal{Y}$ along with the $\mathcal{Y}_{\mathbb{I}}$, $\mathbb{I} \in \mathcal{I}_n$, form a finite clopen cover of $[\emptyset, Y]$. By the Abstract Nash-Williams Theorem, there is $Z \le Y$ such that either $[\emptyset, Z] \subseteq \mathcal{Y}_{\mathbb{I}}$ for some $\mathbb{I} \in \mathcal{I}_n$, or else $[\emptyset, Z] \subseteq \mathcal{Y}$. By Theorem 31, the latter case is impossible. Thus, fix $Z \le Y$ and $\mathbb{I} \in \mathcal{I}_n$ such that $[\emptyset, Z] \subseteq \mathcal{Y}_{\mathbb{I}}$. If at least one of the $I_{n,j}$'s is nonempty then $E_s = E_{\mathbb{I}}$. Otherwise, $E_s = E_{\text{depth}}$. □

The following is obtained from Claim 12 and Lemma 40.

Claim 13 *There is a $B \le A$ such that for each $s \in (\hat{\mathcal{F}} \setminus \mathcal{F})|B$, there is an equivalence relation $E_s \in \mathcal{E}(|s|)$ satisfying the following: For all $a, b \in r_{|s|+1}[s, B]$, B mixes a and b if and only if $a(|s|)\ E_s\ b(|s|)$.*

Fix B as in Claim 13. For $s \in (\hat{\mathcal{F}} \setminus \mathcal{F})|B$ and $n = |s|$, let E_s denote the member of $\mathcal{E}(n)$ as guaranteed by Claim 13. We say that s is *E_s-mixed* by B; that is, for all $a, b \in r_{n+1}[s, B]$, B mixes a and b if and only if $a(n)\ E_s\ b(n)$. Let π_s denote the projection which defines E_s. Given $a \in \mathcal{F}|B$, define

$$\varphi(a) = \bigcup_{i<|a|} \pi_{r_i(a)}(a(i)). \tag{18}$$

The proof of the next claim follows in a straightforward manner from the definitions. We omit the proof, as it is essentially the same as the proof of Claim 4.17 of [11].

Claim 14 *The following are true for all $X \le B$ and all $s, t \in \hat{\mathcal{F}}|B$.*

(1) Suppose $s \notin \mathcal{F}$. Given $a, b \in r_{|s|+1}[s, X]$, if X mixes a and t, and X also mixes b and t, then $a(|s|)\ E_s\ b(|s|)$.
(2) If X separates s and t, then for every $a \in \hat{\mathcal{F}} \cap r_{|s|+1}[s, X]/t$ and every $b \in \mathcal{F} \cap r_{|t|+1}[t, X]/s$, X separates a and b.
(3) Suppose $s \notin \mathcal{F}$. Then $\pi_s = \pi_{<>}$ if and only if X mixes s and a, for all $a \in \hat{\mathcal{F}} \cap r_{|s|+1}[s, X]$.
(4) Suppose $s \notin \mathcal{F}$. Then $\pi_s = \pi_{\text{depth}}$ if and only if for all $a, b \in \hat{\mathcal{F}} \cap r_{|s|+1}[s, X]$, if X mixes a and b then $\text{depth}_X(a) = \text{depth}_X(b)$.
(5) If $s \sqsubseteq t$ and $\varphi(s) = \varphi(t)$, then X mixes s and t.

The next proposition is the crucial step in the proof of the theorem. It follows the same outline as Claim 4.18 of [11], but more needs to be checked for the general setting of topological Ramsey spaces constructed from a generating sequence. The key to this proof is that blocks are composed of sequences of members of Fraïssé classes, and the definition of generating sequence allows us to find blocks where all possible order configurations of some fixed finite collection of structures occur.

Proposition 42 *Assume that $s, t \in (\hat{\mathcal{F}} \setminus \mathcal{F})|B$ are mixed by B. Let $k = |s|$ and $l = |t|$. Then the following hold.*

(a) $\pi_s = \pi_{<>}$ *if and only if* $\pi_t = \pi_{<>}$.
(b) $\pi_s = \pi_{\text{depth}}$ *if and only if* $\pi_t = \pi_{\text{depth}}$.
(c) $\pi_s = \pi_{(I_{s,j})_{j\in J_k}}$ *if and only if* $\pi_t = \pi_{(I_{t,j})_{j\in J_l}}$.

In the case of (c), the set $\{j \in J_k : I_{s,j} \neq \emptyset\}$ *must equal* $\{j \in J_l : I_{t,j} \neq \emptyset\}$*, and the projected substructures are isomorphic. That is, if* $\langle i, (\boldsymbol{S}_{k,j})_{j\in J_k}\rangle = \pi_s(Z(k))$ *and* $\langle i', (\boldsymbol{T}_{|t|,j})_{j\in J_l}\rangle = \pi_t(Z'(l))$*, then for each* $j \in J_k \cap J_l$*, the structures* $\boldsymbol{S}_{k,j}$ *and* $\boldsymbol{T}_{l,j}$ *are isomorphic; in addition, for each* $j \in J_k\setminus J_l$*,* $\boldsymbol{S}_{k,j} = \emptyset$*, and for each* $j \in J_l\setminus J_k$*,* $\boldsymbol{T}_{l,j} = \emptyset$*.*

Furthermore, there is a $C \le B$ *such that for all* $s, t \in \hat{\mathcal{F}}|C$*, if* C *mixes* s *and* t*, then for every* $a \in \hat{\mathcal{F}} \cap r_{k+1}[s, C]/t$ *and every* $b \in \hat{\mathcal{F}} \cap r_{l+1}[t, C]/s$*,* C *mixes* a *and* b *if and only if* $\pi_s(a(k)) = \pi_t(b(l))$*.*

Proof Suppose $s, t \in (\hat{\mathcal{F}}\setminus\mathcal{F})|B$ are mixed by B, and let $k = |s|$ and $l = |t|$.

(a) Suppose $\pi_s = \pi_{<>}$ and $\pi_t \neq \pi_{<>}$. By (1) of Claim 14, B mixes s with at most one E_t equivalence class of extensions of t. Since $\pi_t \neq \pi_{<>}$, there is a $Y \in [\max(k, l), B]$ such that for every $b \in r_{l+1}[t, Y]/s$, Y separates s and b. But then Y separates s and t, contradiction. Thus, π_t must also be $\pi_{<>}$. By a similar argument, we conclude that $\pi_s = \pi_{<>}$ if and only if $\pi_t = \pi_{<>}$.

(b) will follow from the argument for (c), in the case when all $I_{s,j}$ and $I_{t,j}$ are empty.

(c) Suppose now that both π_s and π_t are not $\pi_{<>}$. Let $m = \max(k, l)$, and let $n > m$ be large enough that all amalgamations of two copies of $(\boldsymbol{A}_{m,j})_{j\in J_m}$ can be embedded into $(\boldsymbol{A}_{n,j})_{j\in J_m}$. Let

$$\mathcal{Z}_0 = \{X \in [m, B] : B \text{ separates } s^\frown X(k) \text{ and } t^\frown \pi_{\boldsymbol{A}_l}(X(n))\}$$

and

$$\mathcal{Z}_1 = \{X \in [m, B] : B \text{ separates } s^\frown \pi_{\boldsymbol{A}_k}(X(n)) \text{ and } t^\frown X(l)\}.$$

Applying the Abstract Nash-Williams Theorem twice, we obtain an $X \in [m, B]$ such that $[m, X]$ is homogeneous for both $\mathcal{Z}_0$ and $\mathcal{Z}_1$. Since we are assuming that both π_s and π_t are different from $\pi_{<>}$, it must be the case that $[m, X] \subseteq \mathcal{Z}_0 \cap \mathcal{Z}_1$. Thus, for all $a \in r_{k+1}[s, X]/t$ and $b \in r_{l+1}[t, X]/s$, if a and b are mixed by B, then $\text{depth}_B(a) = \text{depth}_B(b)$.

Let $\mathcal{I}_k$ denote the collection of all sequences of the form $(I_j)_{j\in J_k}$, where each $I_j \subseteq K_{n,j}$ and $\pi_{(I_j)_{j\in J_k}}(B(n)) \in \mathcal{R}(k)$. Likewise, let $\mathcal{I}_l$ denote the collection of all sequences of the form $(I_j)_{j\in J_l}$, where each $I_j \subseteq K_{n,j}$ and $\pi_{(I_j)_{j\in J_l}}(B(n)) \in \mathcal{R}(l)$.

For each pair $\mathbb{S} \in \mathcal{I}_k$ and $\mathbb{T} \in \mathcal{I}_l$, let

$$\mathcal{X}_{\mathbb{S},\mathbb{T}} = \{Y \le X : B \text{ mixes } s^\frown \pi_{\mathbb{S}}(Y(n)) \text{ and } t^\frown \pi_{\mathbb{T}}(Y(n))\}. \tag{19}$$

Applying the Abstract Nash-Williams Theorem finitely-many times, we find a $Y \le X$ which is homogeneous for $\mathcal{X}_{\mathbb{S},\mathbb{T}}$, for all pairs $(\mathbb{S}, \mathbb{T}) \in \mathcal{I}_k \times \mathcal{I}_l$.

Subclaim For each pair $(\mathbb{S}, \mathbb{T}) \in \mathcal{I}_k \times \mathcal{I}_l$, if $\pi_s \circ \pi_{\mathbb{S}}(Y(n)) \neq \pi_t \circ \pi_{\mathbb{T}}(Y(n))$, then $[\emptyset, Y] \cap \mathcal{X}_{\mathbb{S},\mathbb{T}} = \emptyset$.

Proof Suppose $\pi_s \circ \pi_{\mathbb{S}}(Y(n)) \neq \pi_t \circ \pi_{\mathbb{T}}(Y(n))$. Let $S(k)$ denote $\pi_{\mathbb{S}}(Y(n))$ which is in $\mathcal{R}(k)$, and let $T(l)$ denote $\pi_{\mathbb{T}}(Y(n))$ which is in $\mathcal{R}(l)$. Then there is a d and there are some substructures $\boldsymbol{S}_{k,j} \in \binom{\boldsymbol{Y}_{n,j}}{\boldsymbol{A}_{k,j}}$, $j \in J_k$, and $\boldsymbol{T}_{l,j} \in \binom{\boldsymbol{Y}_{n,j}}{\boldsymbol{A}_{l,j}}$, $j \in J_l$ such that $S(k) = \langle d, (\boldsymbol{S}_{k,j})_{j\in J_k}\rangle$ and $T(l) = \langle d, (\boldsymbol{T}_{l,j})_{j\in J_l}\rangle$. $\pi_s \circ \pi_{\mathbb{S}}(Y(n)) = \pi_s(S(k)) = \langle d, (\boldsymbol{S}'_{k,j})_{j\in J_k}\rangle$ for some substructures $\boldsymbol{S}'_{k,j} \leq \boldsymbol{S}_{k,j}$; likewise, $\pi_t \circ \pi_{\mathbb{T}}(Y(n)) = \pi_t(T(l)) = \langle d, (\boldsymbol{T}'_{l,j})_{j\in J_l}\rangle$ for some substructures $\boldsymbol{T}'_{l,j} \leq \boldsymbol{T}_{l,j}$. Since $\pi_s \circ \pi_{\mathbb{S}}(Y(n)) \neq \pi_t \circ \pi_{\mathbb{T}}(Y(n))$, one of the following must happen: (i) there is some $j \in J_k \cap J_l$ such that $\boldsymbol{S}'_{k,j} \neq \boldsymbol{T}'_{k,j}$; or (ii) there is a $j \in J_k \setminus J_l$ such that $\boldsymbol{S}'_{k,j} \neq \emptyset$; or (iii) there is a $j \in J_l \setminus J_k$ such that $\boldsymbol{T}'_{k,j} \neq \emptyset$.

In case (i), without loss of generality, assume that $|\boldsymbol{S}'_{k,j}|\setminus|\boldsymbol{T}'_{l,j}| \neq \emptyset$ for some $j \in J_k \cap J_l$; that is, the universe of $\boldsymbol{S}'_{k,j}$ is not contained within the universe of $\boldsymbol{T}'_{l,j}$. Since $\boldsymbol{S}'_{k,j}$ and $\boldsymbol{T}'_{k,j}$ are substructures of $\boldsymbol{Y}_{n,j}$, their universes are subsets of the universe of $\boldsymbol{Y}_{n,j}$. Recall that $K_{n,j}$ is the cardinality of the universe of $\boldsymbol{Y}_{n,j}$, and that we enumerate the members of the universe $|\boldsymbol{Y}_{n,j}|$ in increasing order as $\{y^i_{n,j} : i \in K_{n,j}\}$. Let $p \in K_{n,j}$ be such that $y^p_{n,j} \in |\boldsymbol{S}'_{k,j}|\setminus|\boldsymbol{T}'_{k,j}|$. Take q large enough that there are $W(n), V(n) \in \mathcal{R}(n)|Y(q)$ such that for all $i \in J_n\setminus\{j\}$, $\boldsymbol{W}_{n,i} = \boldsymbol{V}_{n,i}$, and the universes of $\boldsymbol{W}_{n,j}$ and $\boldsymbol{V}_{n,j}$ differ only on the members $w^p_{n,j}$ and $v^p_{n,j}$. This is possible by the definition of a generating sequence; in particular, because $\mathcal{K}_j$ is a Fraïssé class.

Let $U(k) = \pi_{\mathbb{S}}(W(n))$, $U'(k) = \pi_{\mathbb{S}}(V(n))$, $Z(l) = \pi_{\mathbb{T}}(W(n))$, and $Z'(l) = \pi_{\mathbb{T}}(V(n))$. Then $\pi_t(Z(l)) = \pi_t(Z'(l))$, which implies that B mixes $t^\frown Z(l)$ and $t^\frown Z'(l)$. If $[\emptyset, Y] \subseteq \mathcal{X}_{\mathbb{S},\mathbb{T}}$, then it follows that B mixes $s^\frown U(k)$ and $t^\frown Z(l)$, and B mixes $s^\frown U'(k)$ and $t^\frown Z'(l)$. By transitivity of mixing, B mixes $s^\frown U(k)$ and $s^\frown U'(k)$. But $\pi_s(U(k)) \neq \pi_s(U'(k))$, since $w^p_{n,j} \in \pi_s(U(k))\setminus\pi_s(U'(k))$ (and also $v^p_{n,j} \in \pi_s(U'(k))\setminus\pi_s(U(k))$). Thus, $U(k)$ is not E_s related to $U'(k)$, so B does not mix $s^\frown U(k)$ and $s^\frown U'(k)$, by Claim 13, a contradiction. Therefore, it must be the case that $[\emptyset, Y] \cap \mathcal{X}_{\mathbb{S},\mathbb{T}} = \emptyset$.

In case (ii), if there is a $j \in J_k\setminus J_l$ with $\boldsymbol{S}'_{k,j} \neq \emptyset$, then this implies that $J_k > J_l$. Take $W(n), V(n) \in \mathcal{R}(n)|Y(q)$, for some q large enough, such that $\boldsymbol{W}_{n,j}$ and $\boldsymbol{V}_{n,j}$ have disjoint universes, and for all $i \in J_n\setminus\{j\}$, $\boldsymbol{W}_{n,i} = \boldsymbol{V}_{n,i}$. Let $U(k) = \pi_{\mathbb{S}}(W(n))$, $U'(k) = \pi_{\mathbb{S}}(V(n))$, $Z(l) = \pi_{\mathbb{T}}(W(n))$, and $Z'(l) = \pi_{\mathbb{T}}(V(n))$. Then $Z(l) = Z'(l)$; so in particular, B mixes $t^\frown\pi_t\circ\pi_{\mathbb{T}}(W(n))$ and $t^\frown\pi_t\circ\pi_{\mathbb{T}}(V(n))$. Again, if $[\emptyset, Y] \subseteq \mathcal{X}_{\mathbb{S},\mathbb{T}}$, then B mixes $s^\frown U(k)$ and $t^\frown Z(l)$, and B mixes $s^\frown U'(k)$ and $t^\frown Z'(l)$. By transitivity of mixing, B mixes $s^\frown U(k)$ and $s^\frown U'(k)$. But $\pi_s(U(k)) \neq \pi_s(U'(k))$, since the universes of $\boldsymbol{W}_{n,j}$ and $\boldsymbol{V}_{n,j}$ are disjoint, and the j-th structures in $\pi_s(U(k))$ and $\pi_s(U'(k))$ are not empty and not equal. Thus, B does not mix $s^\frown U(k)$ and $s^\frown U'(k)$, by Claim 13, a contradiction. Therefore, $[\emptyset, Y] \cap \mathcal{X}_{\mathbb{S},\mathbb{T}} = \emptyset$.

By a similar argument as in Case (ii), we conclude that $[\emptyset, Y] \cap \mathcal{X}_{\mathbb{S},\mathbb{T}} = \emptyset$ in Case (iii) as well. Thus, in all cases, $[\emptyset, Y] \cap \mathcal{X}_{\mathbb{S},\mathbb{T}} = \emptyset$. □

It follows from the Subclaim that whenever $a \in r_{k+1}[s, Y]/t$, $b \in r_{l+1}[t, Y]/s$ and B mixes a and b, then $\pi_s(a(k)) = \pi_t(b(l))$. Thus, (b) and (c) follow.

Now we prove there is a $C \leq Y$ such that for all $s, t \in \hat{\mathcal{F}}|C$, if C mixes s and t, then for every $a \in \hat{\mathcal{F}} \cap r_{k+1}[s, C]/t$ and every $b \in \hat{\mathcal{F}} \cap r_{l+1}[t, C]/s$, C mixes a and b if and only if $\pi_s(a(k)) = \pi_t(b(l))$. Since the forward direction holds below Y, it only remains to find a $C \leq Y$ such that, below C, whenever $\pi_s(a(k)) = \pi_t(b(l))$, then C mixes a and b.

Let $(\mathbb{S}, \mathbb{T}) \in \mathcal{I}_k \times \mathcal{I}_l$ be a pair such that $\pi_s \circ \pi_{\mathbb{S}}(Y(n)) = \pi_t \circ \pi_{\mathbb{T}}(Y(n))$. It suffices to show that $[\emptyset, Y] \subseteq \mathcal{X}_{\mathbb{S},\mathbb{T}}$. Assume also, towards a contradiction, that $[\emptyset, Y] \cap \mathcal{X}_{\mathbb{S},\mathbb{T}} = \emptyset$. Let $\mathbb{S}', \mathbb{T}'$ be a pair in $\mathcal{I}_k \times \mathcal{I}_l$ satisfying $\pi_s \circ \pi_{\mathbb{S}'}(Y(n)) = \pi_t \circ \pi_{\mathbb{T}'}(Y(n))$. We will prove that $[\emptyset, Y] \cap \mathcal{X}_{\mathbb{S}',\mathbb{T}'} = \emptyset$. Take $V(n), W(n) \in \mathcal{R}(n)|Y$ such that $\pi_s \circ \pi_{\mathbb{S}}(V(n)) = \pi_s \circ \pi_{\mathbb{S}'}(W(n))$ and $\pi_t \circ \pi_{\mathbb{T}}(V(n)) = \pi_t \circ \pi_{\mathbb{T}'}(W(n))$. This is possible since all $\mathcal{K}_j$ are Fraïssé classes with the OPFAP and we are assuming that $\pi_s \circ \pi_{\mathbb{S}}(Y(n)) = \pi_t \circ \pi_{\mathbb{T}}(Y(n))$ and $\pi_s \circ \pi_{\mathbb{S}'}(Y(n)) = \pi_t \circ \pi_{\mathbb{T}'}(Y(n))$. Then Y mixes $s^\frown\pi_{\mathbb{S}}(V(n))$ and $s^\frown\pi_{\mathbb{S}'}(W(n))$; and Y mixes $t^\frown\pi_{\mathbb{T}}(V(n))$ and $t^\frown\pi_{\mathbb{T}'}(W(n))$. Since Y separates $s^\frown\pi_{\mathbb{S}}(V(n))$ and $t^\frown\pi_{\mathbb{T}}(V(n))$, and since mixing is transitive, it follows that Y must separate $s^\frown\pi_{\mathbb{S}'}(W(n))$ and $t^\frown\pi_{\mathbb{T}'}(W(n))$. Thus, $[\emptyset, Y] \cap \mathcal{X}_{\mathbb{S}',\mathbb{T}'} = \emptyset$.

This, along with the Subclaim, implies that for all pairs $(\mathbb{S}, \mathbb{B})$ in $\mathcal{I}_k \times \mathcal{I}_l$, $[\emptyset, Y] \cap \mathcal{X}_{\mathbb{S},\mathbb{T}} = \emptyset$. But this implies that Y separates s and t, a contradiction. Therefore, for all pairs $(\mathbb{S}, \mathbb{T}) \in \mathcal{I}_k \times \mathcal{I}_l$ such that $\pi_s \circ \pi_{\mathbb{S}}(Y(n)) = \pi_t \circ \pi_{\mathbb{T}}(Y(n))$, we have $[\emptyset, Y] \subseteq \mathcal{X}_{\mathbb{S},\mathbb{T}}$. Thus, whenever $U(k) \in \mathcal{R}(k)|Y/(s,t)$ and $V(l) \in \mathcal{R}(l)|Y/(s,t)$ satisfy $\pi_s(U(k)) = \pi_t(V(l))$, then Y mixes $s^\frown U(k)$ and $t^\frown V(l)$. By Lemma 41, we get $C \leq Y$ for which the Proposition holds. □

By very slight, straightforward modifications to the proofs of Claims 4.19–4.21 in [11], we obtain the following claim.

Claim 15 *For all $a, b \in \mathcal{F}|C$, $a\,R\,b$ if and only if $\varphi(a) = \varphi(b)$. Moreover, $\varphi(a) \not\sqsubset \varphi(b)$.*

By its definition, φ is inner, and by Claim 15, φ is Nash-Williams and canonizes the equivalence relation R. □

Remark We point out that the entire proof of Theorem 38 used only instances of the Abstract Nash-Williams Theorem, and not the full power of the Abstract Ellentuck Theorem.

The following corollary of Theorem 38 is proved in exactly the same way as Theorem 4.3 in [11].

Corollary 43 *Let $1 \leq J \leq \omega$, and $\mathcal{K}_j$, $j \in J$, be Fraïssé classes of finite ordered relational structures with the Ramsey property and the Order-Prescribed Free Amalgamation Property. Let $\langle \mathbf{A}_k : k < \omega \rangle$ be a fixed generating sequence, and let $\mathcal{R}$ denote the topological Ramsey space $\mathcal{R}(\langle \mathbf{A}_k : k < \omega \rangle)$.*

For any n, $A \in \mathcal{R}$, and equivalence relation R on $\mathcal{AR}_n|A$, there is a $B \leq A$ such that R is canonical on $\mathcal{AR}_n|B$. This means there are equivalence relations $E_i \in \mathcal{E}(i)$, $i < n$, such that for all $a, b \in \mathcal{AR}_n|B$

$$a\,R\,b \text{ if and only if } \forall i < n, a(i)\; E_i\; b(i).$$

7 Basic Tukey reductions for selective and Ramsey filters on general topological Ramsey spaces

We first remind the reader of the basic definitions of the Tukey theory of ultrafilters. Suppose that $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters. A function f from $\mathcal{U}$ to $\mathcal{V}$ is *cofinal* if every cofinal subset of $(\mathcal{U}, \supseteq)$ is mapped by f to a cofinal subset of $(\mathcal{V}, \supseteq)$. We say that $\mathcal{V}$ is *Tukey reducible to* $\mathcal{U}$ and write $\mathcal{V} \leq_T \mathcal{U}$ if there exists a cofinal map $f : \mathcal{U} \to \mathcal{V}$. If $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$ then we write $\mathcal{V} \equiv_T \mathcal{U}$ and say that $\mathcal{U}$ and $\mathcal{V}$ are *Tukey equivalent*. The relation $\equiv_T$ is an equivalence relation and $\leq_T$ is a partial order on its equivalence classes. The equivalence classes are called *Tukey types*. (See the recent survey paper [7] for more background on Tukey types of ultrafilters.)

When restricted to ultrafilters, the Tukey reducibility relation is a coarsening of the Rudin-Keisler reducibility relation. If $h(\mathcal{U}) = \mathcal{V}$, then the map sending $X \in \mathcal{U}$ to $h''X \in \mathcal{V}$ witnesses Tukey reducibility. Thus, if $\mathcal{V} \leq_{RK} \mathcal{U}$, then $\mathcal{V} \leq_T \mathcal{U}$.

The work in this section will set up some of the machinery for answering this and Questions 1, 2, and 3 from the Introduction; we do that in the next section. The main results in this section, Proposition 55 and Theorem 56, are proved for general topological Ramsey spaces, in the hope that they may be more generally applied in the future.

An ultrafilter $\mathcal{U}$ on a countable base X has *continuous Tukey reductions* if whenever a non-principal ultrafilter $\mathcal{V}$ is Tukey reducible to $\mathcal{U}$, then every monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$ is continuous with respect to the subspace topologies on $\mathcal{U}$ and $\mathcal{V}$ inherited from 2^X when restricted to some cofinal subset of $\mathcal{U}$. The next theorem has become an important tool in the study of the Tukey structure of ultrafilters Tukey reducible to some p-point ultrafilter.

Theorem 44 ([10]) *If $\mathcal{U}$ is a p-point ultrafilter on ω, then $\mathcal{U}$ has continuous Tukey reductions.*

In fact, by results of Dobrinen (see Theorem 2.7 in [6], which first appeared in the unpublished [5]), every ultrafilter Tukey reducible to some p-point has continuous Tukey reductions.

In the previous sections of this paper we restricted consideration to spaces constructed from generating sequences. In this section we consider all topological Ramsey spaces $\mathcal{R}$ such that $\mathcal{R}$ is closed in $\mathcal{AR}^{\mathbb{N}}$, $(\mathcal{R}, \leq)$ is a partial order, and $(\mathcal{R}, \leq, r)$ satisfies axioms **A.1** – **A.4**. In Theorem 56, we generalize Theorem 44 to filters selective for a topological Ramsey space.

Notation In order to avoid repeating certain phrases, we let $(\mathcal{R}, \leq, r)$ denote a fixed triple satisfying axioms **A.1** – **A.4** which is closed in the subspace topology it inherits from $\mathcal{AR}^{\mathbb{N}}$. Furthermore, we assume that $(\mathcal{R}, \leq)$ is a partial order and has a top element which we denote by $\mathbb{A}$. By the Abstract Ellentuck Theorem, $\mathcal{R}$ forms a topological Ramsey space. If $\mathcal{C}$ is a subset of $\mathcal{R}$ we let $\mathcal{C} \geq_T \mathcal{V}$ denote the statement $(\mathcal{C}, \geq) \geq_T (\mathcal{V}, \supseteq)$.

We omit the proof of the next fact since if follows by a straightforward generalization of the proof of Fact 6 from [10].

Fact 45 *Assume that $\mathcal{C} \subseteq \mathcal{R}$ and $\mathcal{V}$ is an ultrafilter on ω. If $\mathcal{C} \geq_T \mathcal{V}$, then there is monotone cofinal map $f : \mathcal{C} \to \mathcal{V}$.*

The notion of a selective filter for a topological Ramsey space was introduced along with the relation of almost-reduction by Mijares [18]. The notion of almost reduction on a topological Ramsey space was introduced by Mijares [18]. The relation of almost reduction generalizes the relation of almost inclusion $\subseteq^*$ on $\mathcal{P}(\omega)$ to arbitrary topological Ramsey spaces. The relation of *almost reduction on* $\mathcal{R}$ is defined as follows: $X \leq^* Y$ if and only if there exists $a \in \mathcal{AR}$ such that $\emptyset \neq [a, X] \subseteq [a, Y]$. Fix the following notation: For any fixed $\mathbb{A} \in \mathcal{R}$, for $n < \omega$ and $X, Y \leq \mathbb{A}$, define $X/n \leq Y$ if and only if there exists an $a \in \mathcal{AR}|Y$ with $\text{depth}_{\mathbb{A}}(a) \leq n$, $\emptyset \neq [a, X] \subseteq [a, Y]$. In particular, if $X/n \leq Y$, then $X \leq^* Y$. If a topological Ramsey space has a maximum member, we let $\mathbb{A}$ denote that member. Otherwise, we may without loss of generality fix some $\mathbb{A} \in \mathcal{R}$ and work below $\mathbb{A}$.

Fact 46 *For each X and Y in $\mathcal{R}$, $X \leq^* Y$ if and only if there exists $i < \omega$ such that $X/r_i(X) \leq Y$.*

Definition 47 *A subset $\mathcal{C} \subseteq \mathcal{R}$ is a* selective filter *on $(\mathcal{R}, \leq)$ if $\mathcal{C}$ is a maximal filter on $(\mathcal{R}, \leq)$ and for each decreasing sequence $X_0 \geq X_1 \geq X_2 \geq \dots$ of elements of $\mathcal{C}$ there exists $X \in \mathcal{C}$ such that for all $i < \omega$, $X/r_i(X) \leq X_i$.*

Axiom **A.3** implies that for each decreasing sequence $X_0 \geq X_1 \geq X_2 \geq \dots$ of elements of $\mathcal{R}$ there exists $X \in \mathcal{R}$ such that for all $i < \omega$, $X/r_i(X) \leq X_i$. Thus, assuming MA or CH it is possible to construct a selective filter on $(\mathcal{R}, \leq)$. Forcing with $\mathcal{R}$ using almost reduction adjoins a filter on $(\mathcal{R}, \leq)$ satisfying a localized version of the Abstract Nash-Williams theorem for $\mathcal{R}$. By work of Mijares [18] every ultrafilter generic for this forcing is a selective filter on $(\mathcal{R}, \leq)$.

Recall that $\mathcal{R}$ is assumed to be closed in the subspace topology it inherits from $\mathcal{AR}^{\mathbb{N}}$. A sequence $(X_n)_{n<\omega}$ of elements of $\mathcal{R}$ converges to an element $X \in \mathcal{R}$ if and only if for each $k < \omega$ there is an $m < \omega$ such that for each $n \geq m$, $r_k(X_n) = r_k(X)$. A function $f : \mathcal{R} \to \mathcal{P}(\omega)$ is *continuous* if and only if for each convergent sequence $(X_n)_{n<\omega}$ in $\mathcal{R}$ with $X_n \to X$, we also have $f(X_n) \to f(X)$ in the topology obtained by identifying $\mathcal{P}(\omega)$ with $2^{\mathbb{N}}$. A function $f : \mathcal{C} \to \mathcal{V}$ is said to be *continuous* if it is continuous with respect to the topologies on $\mathcal{C}$ and $\mathcal{V}$ taken as subspaces of $\mathcal{AR}^{\mathbb{N}}$ and $2^{\mathbb{N}}$, respectively. The next definition is a generalization of notion of basic Tukey reductions for an ultrafilter on ω, (see Definition 2.2 and Lemma 2.5 in [6]), to filters on $\mathcal{R}$.

Definition 48 *Assume that $\mathcal{C} \subseteq \mathcal{R}$ is a filter on $(\mathcal{R}, \leq)$. $\mathcal{C}$* has basic Tukey reductions *if whenever $\mathcal{V}$ is a non-principal ultrafilter on ω and $f : \mathcal{C} \to \mathcal{V}$ is a monotone cofinal map, there is an $X \in \mathcal{C}$ and a monotone map $\tilde{f} : \mathcal{R} \to \mathcal{P}(\omega)$ such that*

(1) $\tilde{f}$ *is continuous with respect to the metric topology on* $\mathcal{AR}^{\mathbb{N}}$*;*
(2) $\tilde{f} \upharpoonright (\mathcal{C} \upharpoonright X) = f \upharpoonright (\mathcal{C} \upharpoonright X)$*;*
(3) $\tilde{f}$ *is generated by a finitary map* $\hat{f} : \mathcal{AR} \to [\omega]^{<\omega}$ *satisfying*
   (a) *For each $k < \omega$ and each $s \in \mathcal{AR}$, if* $\text{depth}_{\mathbb{A}}(s) \leq k$ *then* $\hat{f}(s) \subseteq k$;
   (b) $s \sqsubseteq t \in \mathcal{AR}$ *implies that* $\hat{f}(s) \sqsubseteq \hat{f}(t)$*;*

(c) $\hat{f}$ *is monotone, that is, if* $s, t \in \mathcal{AR}$ *with* $s \leq_{\text{fin}} t$, *then* $\hat{f}(s) \subseteq \hat{f}(t)$; *and*
(d) *For each* $Y \in \mathcal{R}$, $\tilde{f}(Y) = \bigcup_{k<\omega} \hat{f}(r_k(Y))$.

The next proposition provides an important application of the notion of basic Tukey reductions for $\mathcal{C}$ and helps reduce the characterization of the ultrafilters on ω Tukey reducible to $(\mathcal{C}, \geq)$ to the study of canonical equivalence relations for fronts on $\mathcal{C}$. It is the generalization of Proposition 5.5 from [11] to our current setting.

Definition 49 *If* $\mathcal{C} \subseteq \mathcal{R}$ *and* $\mathcal{F} \subseteq \mathcal{AR}$ *then we will say that* $\mathcal{F}$ is a front on $\mathcal{C}$ *if and only if for each* $C \in \mathcal{C}$, *there exists* $s \in \mathcal{F}$ *such that* $s \sqsubseteq X$; *and for all pairs* $s \neq t$ *in* $\mathcal{F}$, $s \not\sqsubset t$.

Proposition 50 *Assume that* $\mathcal{C} \subseteq \mathcal{R}$ *is a filter on* $(\mathcal{R}, \leq)$ *which has basic Tukey reductions, and suppose* $\mathcal{V}$ *is a non-principal ultrafilter on* ω *with* $\mathcal{V} \leq_T \mathcal{C}$. *Then there is a front* $\mathcal{F}$ *on* $\mathcal{C}$ *and a function* $f : \mathcal{F} \to \omega$ *such that for each* $Y \in \mathcal{V}$, *there exists* $X \in \mathcal{C}$ *such that* $f(\mathcal{F}|X) \subseteq Y$. *Furthermore, if* $\mathcal{C} \upharpoonright \mathcal{F}$ *is a base for an ultrafilter on* $\mathcal{F}$, *then* $\mathcal{V} = f(\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle)$.

Proof Suppose that $\mathcal{C}$ and $\mathcal{V}$ are given and satisfy the assumptions of the proposition. By Fact 45, there is a monotone map $g : \mathcal{C} \to \mathcal{V}$. Since $\mathcal{C}$ has basic Tukey reductions, there is a continuous monotone cofinal map $g' : \mathcal{C} \to \mathcal{V}$ and a function $\hat{g} : \mathcal{AR} \to [\omega]^{<\omega}$ satisfying (1)-(3) in the definition of basic Tukey reductions. Let $\mathcal{F}$ consist of all $r_n(Y)$ such that $Y \in \mathcal{C}$ and n is minimal such that $\hat{g}(r_n(Y)) \neq \emptyset$. By the properties of $\hat{g}$, $\min(\hat{g}(r_n(Y))) = \min(g(Y))$. By its definition $\mathcal{F}$ is a front on $\mathcal{C}$. Define a new function $f : \mathcal{F} \to \omega$ by $f(b) = \min(\hat{g}(b))$, for each $b \in \mathcal{F}$.

Since g' is a monotone cofinal map, the g'-image of $\mathcal{C}$ in $\mathcal{V}$ is a base for $\mathcal{V}$. From the construction of f, we see that for each $X \in \mathcal{C}$, $f(\mathcal{F}|X) = \{f(a) : a \in \mathcal{F}|X\} \subseteq g'(X)$. Therefore, for each $Y \in \mathcal{V}$ there exists $X \in \mathcal{C}$ such that $f(\mathcal{F}|X) \subseteq Y$. We remind the reader of the following useful fact (see Fact 5.4 from [11]).

Fact 51 *Suppose* $\mathcal{V}$ *and* $\mathcal{U}$ *are proper ultrafilters on the same countable base set, and for each* $V \in \mathcal{V}$ *there is a* $U \in \mathcal{U}$ *such that* $U \subseteq V$. *Then* $\mathcal{U} = \mathcal{V}$.

Suppose that $\mathcal{C} \upharpoonright \mathcal{F}$ generates an ultrafilter on $\mathcal{F}$, and let $\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle$ denote the ultrafilter it generates. Then the Rudin-Keisler image $f(\langle \mathcal{F} \upharpoonright \mathcal{C} \rangle)$ is an ultrafilter on ω generated by the base $\{f(\mathcal{F}|X) : X \in \mathcal{X}\}$. Hence, Fact 51 implies that $f(\langle \mathcal{F} \upharpoonright \mathcal{C} \rangle) = \mathcal{V}$. □

If a selective filter $\mathcal{C}$ on $(\mathcal{R}, \leq)$ has the property that, for each front $\mathcal{F}$ on $\mathcal{C}$, $\mathcal{C} \upharpoonright \mathcal{F}$ generates an ultrafilter on $\mathcal{F}$, then Proposition 50 shows that every nonprincipal ultrafilter Tukey-reducible to $\mathcal{C}$ is a Rudin-Keisler image of $\mathcal{C} \upharpoonright \mathcal{F}$, for some front on $\mathcal{C}$. This provides motivation for studying the notion of a Nash-Williams filter on $(\mathcal{R}, \leq)$. The next definition is an adaptation of Definition 5.1 (1) from [11] to our current setting.

Definition 52 *A maximal filter* $\mathcal{C} \subseteq \mathcal{R}$ *is a* Nash-Williams filter on $(\mathcal{R}, \leq)$ *if for each front* $\mathcal{F}$ *on* $\mathcal{C}$ *and each* $\mathcal{H} \subseteq \mathcal{F}$, *there is a* $C \in \mathcal{C}$ *such that either* $\mathcal{F}|C \subseteq \mathcal{H}$ *or else* $\mathcal{F}|C \cap \mathcal{H} = \emptyset$.

It is clear that any Nash-Williams filter is also a Ramsey filter on $(\mathcal{R}, \leq)$ (recall Definition 19), and hence is a maximal filter. The Abstract Nash-Williams Theorem for $\mathcal{R}$ can be used in conjunction with MA or CH to construct a Nash-Williams filter on $(\mathcal{R}, \leq)$. Furthermore, forcing with $\mathcal{R}$ using almost reduction adjoins a Nash-Williams filter on $(\mathcal{R}, \leq)$. By work of Mijares [18], any Ramsey filter on $(\mathcal{R}, \leq)$ is a selective filter on $(\mathcal{R}, \leq)$. Thus, any Nash-Williams filter is a selective filter on $(\mathcal{R}, \leq)$. Trujillo [28] has shown that (assuming CH or MA, or by forcing) there are topological Ramsey spaces for which there are maximal filters which are selective but not Ramsey for those spaces. We omit the proof of the next theorem as it follows from a straightforward generalization of the proof of Trujillo [27] for the special case of the space $\mathcal{R}_1$ (recall Example 21 in Sect. 4).

Theorem 53 *Let $1 \leq J \leq \omega$ and $\mathcal{K}_j$, $j \in J$, be a collection of Fraïssé classes of finite ordered relational structures such that each $\mathcal{K}_j$ satisfies the Ramsey property. Let $\langle \mathbf{A}_k : k < \omega \rangle$ be a generating sequence, and let $\mathcal{R}$ denote $\mathcal{R}(\langle \mathbf{A}_k : k < \omega \rangle)$. Suppose that $\mathcal{C}$ is a filter on $(\mathcal{R}, \leq)$. Then $\mathcal{C}$ is Nash-Williams for $\mathcal{R}$ if and only if $\mathcal{C}$ is Ramsey for $\mathcal{R}$.*

The next fact is the analogue of Fact 5.3 from [11]. We omit the proof as it follows by similar arguments.

Fact 54 *Suppose $\mathcal{C} \subseteq \mathcal{R}$ is a Nash-Williams filter on $(\mathcal{R}, \leq)$. If $\mathcal{C}'$ is any cofinal subset of $\mathcal{C}$, and $\mathcal{F} \subseteq \mathcal{AR}$ is any front on $\mathcal{C}'$, then $\mathcal{C}' \restriction \mathcal{F}$ generates an ultrafilter on $\mathcal{F}$.*

The next proposition is one of the keys in the general mechanism for classifying initial Tukey structures and the Rudin-Keisler structures within them. We only sketch the proof here, as it is the same proof as that of Proposition 5.5 in [11].

Proposition 55 *Assume that that $\mathcal{C} \subseteq \mathcal{R}$ is a Nash-Williams filter on $(\mathcal{R}, \leq)$. Suppose $\mathcal{C}$ has basic Tukey reductions and $\mathcal{V}$ is a non-principal an ultrafilter on ω with $\mathcal{C} \geq_T \mathcal{V}$. Then there is a front $\mathcal{F}$ on $\mathcal{C}$ and a function $f : \mathcal{F} \to \omega$ such that $\mathcal{V} = f(\langle \mathcal{C} \restriction \mathcal{F} \rangle)$.*

Proof Suppose that $\mathcal{V}$ is Tukey reducible to some Nash-Williams filter $\mathcal{C}$ on $(\mathcal{R}, \leq)$. Assume that $\mathcal{C}$ has Basic Tukey reductions. Theorem 50 and Fact 54 imply that there is a front $\mathcal{F}$ on $\mathcal{C}$ and a function $f : \mathcal{F} \to \omega$ such that $\mathcal{V} = f(\langle \mathcal{C} \restriction \mathcal{F} \rangle)$. □

We now introduce some notation needed for its definition and for the proof of the main theorem of this section.

Notation If there is a maximum member of $\mathcal{R}$, let $\mathbb{A}$ denote it. Otherwise, fix some $\mathbb{A} \in \mathcal{R}$ and relative everything that follows to $[0, \mathbb{A}]$. For each $X, Y \leq \mathbb{A}$, define

$$d(X) = \{\text{depth}_{\mathbb{A}}(r_i(X)) : i < \omega\}. \tag{20}$$

Define $\rho : [0, \mathbb{A}] \times \omega \to \mathcal{AR}$ to be the map such that for each $X \leq \mathbb{A}$ and each $n < \omega$, $\rho(X, n) = r_i(X)$, where i is the unique natural number such that $\text{depth}_{\mathbb{A}}(r_i(X)) \leq n < \text{depth}_{\mathbb{A}}(r_{i+1}(X))$.

By **A.1** (c) and **A.2** (b), for each $X \le \mathbb{A}$, $d(X)$ is infinite. Also note that for each $s \in \mathcal{AR}|X$, $X/s \le Y$ if and only if $\text{depth}_X(s) = i$ and $X/r_i(X) \le Y$. In particular, $X/r_i(X) \le Y$ if and only if $X/\,\text{depth}_{\mathbb{A}}(r_i(X)) \le Y$.

The next theorem is the main result of this section. It extends to all topological Ramsey spaces previous results in [10] for the Milliken space $\text{FIN}^{[\infty]}$ and in [11] and [12] for the $\mathcal{R}_\alpha$ spaces ($1 \le \alpha < \omega_1$). It will be used in conjunction with Proposition 55 in the next section to identify initial structures in the Tukey types of ultrafilters.

Theorem 56 *If $\mathcal{C}$ is a selective filter on $(\mathcal{R}, \le)$ and $\{d(X) : X \in \mathcal{C}\}$ generates a nonprincipal ultrafilter on ω then, $\mathcal{C}$ has basic Tukey reductions.*

Proof Suppose that $\mathcal{V}$ is an ultrafilter on ω Tukey reducible to $\mathcal{C}$, and $f : \mathcal{C} \to \mathcal{V}$ is a monotone cofinal map witnessing $\mathcal{C} \ge_T \mathcal{V}$. For each $k < \omega$, let $P_k(\cdot, \cdot)$ be the following proposition: For $s \in \mathcal{AR}$ and $X \in \mathcal{R}$, $P_k(s, X)$ holds if and only if for each $Z \in \mathcal{C}$ such that $s \sqsubseteq Z$ and $Z/s \le X$, $k \notin f(Z)$. Let $\mathcal{C}$ be a selective filter for $(\mathcal{R}, \le)$. Assume that $\{d(X) : X \in \mathcal{C}\}$ generates an nonprincipal ultrafilter on ω.

Claim 16 *There is an $\bar{X} \in \mathcal{C}$ such that $f \restriction (\mathcal{C} \restriction \bar{X}) : \mathcal{C} \restriction \bar{X} \to \mathcal{V}$ is continuous.*

Proof We begin by constructing a decreasing sequence in $(\mathcal{C}, \le)$. Let $X_0 = \mathbb{A}$. Given $n > 0$ and $X_i \in \mathcal{C}$ for all $i < n$, we will choose $X_n \in \mathcal{C}$ such that

(1) $X_n \le X_{n-1}$,
(2) $\rho(X_n, n) = \emptyset$,
(3) For each s in $\mathcal{AR}$ with $\text{depth}_{\mathbb{A}}(s) \le n$ and each $k \le n$, if there exists $Y' \in \mathcal{C}$ such that $\rho(Y', n) = s$ and $k \notin f(Y')$, then $P_k(s, X_n)$ holds.

By axiom **A.2** (a), the set $\{s \in \mathcal{AR} : \text{depth}_{\mathbb{A}}(s) \le n\}$ is finite. Let $s_1, s_2, \dots s_{i_n}$ be an enumeration of $\{s \in \mathcal{AR} : \text{depth}_{\mathbb{A}}(s) \le n\}$. Since $\mathcal{C}$ is a maximal filter and $\{d(X) : X \in \mathcal{C}\}$ forms a nonprincipal ultrafilter on ω, there exists a $W_0 \in \mathcal{C}$ such that $W_0 \le X_{n-1}$ and $\rho(W_0, n) = \emptyset$. Now suppose that there exists $Y \in \mathcal{C}$ such that $\rho(Y, n) = s_1$ and $k \notin f(Y)$. Take Y_1 to be in $\mathcal{C}$ such that $\rho(Y_1, n) = s_1$ and $k \notin f(Y_1)$. Since $\mathcal{C}$ is a filter on $(\mathcal{C}, \le)$ there exists $W_1 \in \mathcal{C}$ such that $W_1 \le Y_1, W_0$. If there is no $Y \in \mathcal{C}$ such that $\rho(X, n) = s_1$ and $k \notin f(Y)$, then let $W_1 = W_0$. For the induction step, suppose that for $1 \le l < i_n$ and $W_0 \ge W_1 \ge \cdots \ge W_l$ are given and in $\mathcal{C}$. If there is a $Y \in \mathcal{C}$ such that $s_{l+1} = \rho(Y, n)$ and $k \notin f(Y)$, then take some $Y_l \in \mathcal{C}$ and let $W_{l+1} \in \mathcal{C}$ such that $W_{l+1} \le W_l, Y_{l+1}$. Otherwise, let $W_{l+1} = W_l$. After i_n many steps let $X_n = W_{i_n}$.

We check that X_n satisfies properties (1)–(3). (1) By construction $X_n \le X_{n-1}$. (2) Since $\rho(W_0, n) = \emptyset$ and $X_n \le W_0$, we have $\rho(X_n, n) = \emptyset$. (3) Let s be an element of $\mathcal{AR}$ such that $\text{depth}_{\mathbb{A}}(s) \le n$. It follows that there exists $1 \le l \le i_n$ such that $s = s_l$. If there is a $Y' \in \mathcal{C}$ such that $s = s_l = \rho(Y', n)$ and $k \notin f(Y')$ then W_l was taken so that $W_l \le W_{l-1}, Y'_l$. Hence, if $Z \in \mathcal{C}$, $s \sqsubseteq Z$ and $Z/s \le X_n$ then $Z \le Y'_l$. Since f is monotone and $k \notin f(Y'_l)$, it must be the case that $P_k(s, X_n)$ holds.

Since $\mathcal{C}$ is selective for $(\mathcal{R}, \le)$, there exists $Y \in \mathcal{C}$ such that for each $i < \omega$, $Y/r_i(Y) \le X_i$. Let $\{y_0, y_1, \dots\}$ denote the increasing enumeration of $d(Y)$. Let $A = \bigcup [y_{2i+1}, y_{2i+2})$. Without loss of generality, assume that A is not in the ultrafilter generated by $\{d(X) : X \in \mathcal{C}\}$. Let $\bar{X}$ be an element of $\mathcal{C}$ such that $\bar{X} \le Y$ and

$d(\bar{X}) \subseteq \omega \backslash A$. We show that $f \restriction (\mathcal{C} \restriction \bar{X})$ is continuous by showing that there is a strictly increasing sequence $(m_k)_{k<\omega}$ such that for each $Z \in \mathcal{C} \restriction \bar{X}$, the initial segment $f(Z) \cap (k+1)$ of $f(Z)$ is determined by the initial segment $\rho(Z, m_k)$ of Z.

For each $k < \omega$, let i_k denote the least i for which $y_{2i+1} \geq k$. Let $W \in \mathcal{C} \restriction \bar{X}$ be given and let $s = \rho(W, y_{2i_k+1})$. Since $d(\bar{X}) \cap [y_{2i_k+1}, y_{2i_k+2}) = \emptyset$, it follows that $\rho(\bar{X}, y_{2i_k+1}) = \rho(\bar{X}, y_{2i_k+2})$. Notice that $W \leq \bar{X}$, $\bar{X}/y_{2i_k+2} \leq Y$, $Y/y_{2i_k+2} \leq X_{2i_k+1}$ and $\rho(\bar{X}, y_{2i_k+1}) = \rho(\bar{X}, y_{2i_k+2})$ From this it follows that $k \notin f(W)$ if and only if $P_k(s, X_{2i_k+1})$, which holds if and only if $P_k(s, \bar{X})$ holds. Let $m_k = y_{2i_k+2}$. Then $f \restriction (\mathcal{C} \restriction \bar{X})$ is continuous, since the question of whether or not $k \in f(W)$ is determined by the finite initial segment $\rho(W, m_k)$ along with $\bar{X}$. □

Extend $f \restriction (\mathcal{C} \restriction \bar{X})$ to a function $f' : \mathcal{C} \to \mathcal{V}$ by defining $f'(X) = \bigcup\{f(Y) : Y \in \mathcal{C} \restriction \bar{X}$ and $Y \leq X\}$, for $X \in \mathcal{C}$. Notice that $f' : \mathcal{C} \to \mathcal{V}$ is monotone and $f' \restriction (\mathcal{C} \restriction \bar{X}) = f \restriction (\mathcal{C} \restriction \bar{X})$. Further, for each $X \in \mathcal{C}$ and $k < \omega$, $k \notin f'(X)$ if and only if for all $Y \in \mathcal{C} \restriction \bar{X}$ with $Y \leq X$, $k \notin f'(X)$, and this holds if and only if $P_k(\rho(X, m_k), \bar{X})$ holds. Thus, $f' : \mathcal{C} \to \mathcal{V}$ is continuous, as whether $k \in f(X)$ is determined by the finite initial segment $\rho(X, m_k)$ along with $\bar{X}$. Now define $\hat{f} : \mathcal{AR} \to [\omega]^{<\omega}$ by $\check{f}(s) = \{k \leq \mathrm{depth}_{\mathbb{A}}(s) : \neg P_k(s, \bar{X})\}$, for $s \in \mathcal{AR}$; and define $\tilde{f} : \mathcal{R} \to \mathcal{P}(\omega)$ by $\tilde{f}(Y) = \bigcup_{n<\omega} \hat{f}(r_n(Y))$, for $Y \in \mathcal{R}$. Then $\hat{f}$ satisfies (3) in Definition 48 and $\tilde{f}$ is continuous. Notice that $f'(Y) = \bigcup_{n<\omega} \hat{f}(r_n(Y))$, for $Y \in \mathcal{C}$, hence implying that $\tilde{f} \restriction \mathcal{C} = f' \restriction \mathcal{C}$. Thus, $\tilde{f} \restriction (\mathcal{C} \restriction \bar{X}) = f' \restriction (\mathcal{C} \restriction \bar{X})$. □

Corollary 57 *Let $\langle \mathbf{A}_k : k < \omega \rangle$ be a generating sequence as in Definition 15. If $\mathcal{C}$ is a selective filter on $(\mathcal{R}(\langle \mathbf{A}_k : k < \omega \rangle), \leq)$ such that $\{d(X) : X \in \mathcal{C}\}$ generates a nonprincipal ultrafilter on ω, then for each ultrafilter $\mathcal{V}$ Tukey reducible to $\mathcal{C}$, $\mathcal{V}$ has basic Tukey reductions.*

Proof This follows from Theorem 56 and the proof of Theorem 2.6 in [6]. □

The next result will be used in Section 8 to identify initial structures in the Tukey types of p-point ultrafilters.

Theorem 58 *Suppose $\mathcal{C} \subseteq \mathcal{R}$ is a Nash-Williams filter on $(\mathcal{R}, \leq)$ and $\{d(X) : X \in \mathcal{C}\}$ generates an ultrafilter on ω. Then an ultrafilter $\mathcal{V}$ on ω is Tukey reducible to $\mathcal{C}$ if and only if $\mathcal{V} = f(\langle \mathcal{C} \restriction \mathcal{F} \rangle)$ for some front $\mathcal{F}$ on $\mathcal{C}$ and some function $f : \mathcal{F} \to \omega$.*

Proof ($\Rightarrow$) Suppose that $\mathcal{C}$ is Ramsey for $\mathcal{R}$. Proposition 55 and Theorem 56 show that if $\mathcal{V}$ is a non-principal ultrafilter on ω Tukey reducible to $\mathcal{C}$ then there is a front $\mathcal{F}$ on $\mathcal{C}$ and a function $f : \mathcal{F} \to \omega$ such that $\mathcal{V} = f(\langle \mathcal{C} \restriction \mathcal{F} \rangle)$.

($\Leftarrow$) Suppose that $\mathcal{F}$ is a front on $\mathcal{C}$, $f : \mathcal{F} \to \omega$ and $\mathcal{V} = f(\langle \mathcal{C} \restriction \mathcal{F} \rangle)$. The map sending $X \in \mathcal{C}$ to $f''\mathcal{F}|X$ is a monotone cofinal map from $(\mathcal{C}, \geq)$ to $(\mathcal{V}, \supseteq)$. Thus, $\mathcal{V} \leq_T \mathcal{C}$. □

When $\mathcal{R}$ is a topological Ramsey space constructed from a generating sequence, Theorem 53 implies that the hypotheses of Theorem 58 can be weakened to assuming that $\mathcal{C}$ is Ramsey.

The next fact shows that many topological Ramsey spaces give rise to selective filters with basic Tukey reductions.

Fact 59 *Suppose that $\mathcal{R}$ has the property that for each $X \in \mathcal{R}$ and each $A \subseteq \omega$ there exists $Y \leq X$ in $\mathcal{R}$ such that either $d(Y) \subseteq A$ or $d(Y) \subseteq \omega \setminus A$. Then assuming CH, MA or forcing with $\mathcal{R}$ using almost reduction, there exists a selective filter on $(\mathcal{R}, \leq)$ with the property that $\{d(X) : X \in \mathcal{C}\}$ generates a nonprincipal ultrafilter on ω.*

Dobrinen and Todorcevic [11,12] introduced topological Ramsey spaces $\mathcal{R}_\alpha$, $\alpha < \omega_1$, which distill key properties of forcings of Laflamme [17] and with associated ultrafilters with initial Tukey structure exactly that of a decreasing chain of order type $\alpha + 1$. For $1 \leq n < \omega$, the space $\mathcal{R}_n$ is constructed from a certain tree of height $n + 1$ which forms the top element of the space. When $n > 1$, these spaces are not constructed from generating sequences.

Trujillo has shown in [28] that there is a topological Ramsey space $\mathcal{R}_n^\star$ constructed from $\mathcal{R}_n$, such that forcing with $\mathcal{R}_n^\star$ using almost reduction adjoins a selective filter $\mathcal{C}$ on $(\mathcal{R}_n, \leq)$ which is not a Ramsey filter on $(\mathcal{R}_n, \leq)$. Furthermore, it can be shown that $\{d(X) : X \in \mathcal{C}\}$ generates an ultrafilter on ω. Forcing with the space $\mathcal{R}_n^\star$ using almost reduction, or assuming CH or MA, one can construct a selective but not Ramsey maximal filter on $(\mathcal{R}_n, \leq)$. Such a filter has the property that $\{d(X) : X \in \mathcal{C}\}$ generates an ultrafilter on ω. Theorem 56 implies that these non-Ramsey filters on $(\mathcal{R}_n, \leq)$ have basic Tukey reductions. Using a similar argument, the work of Trujillo [28] shows that for each positive n, using forcing or assuming CH or MA, there is a selective but not Ramsey filter on $(\mathcal{H}^n, \leq)$ with basic Tukey reductions. (Recall $\mathcal{H}^n$ from Example 24.)

If $\mathcal{R}$ is constructed from some generating sequence then Theorems 56 and 58 reduce the identification of ultrafilters on ω which are Tukey reducible to a Ramsey filter $\mathcal{C}$ associated with $(\mathcal{R}, \leq)$ to the study of Rudin-Keisler reduction on ultrafilters on base sets which are fronts on $\mathcal{C}$. In the next section we show that the Ramsey-classification Theorem 38 can be localized to equivalence relations on fronts on a Ramsey filter on $(\mathcal{C}, \leq)$. We then use it identify initial structures in the Tukey types of ultrafilters Tukey reducible to any Ramsey filter associated with a Ramsey space constructed from a generating sequence.

8 Initial structures in the Tukey and Rudin-Keisler types of p-points

The structure of the Tukey types of ultrafilters (partially ordered by $\supseteq$) was studied in [10]. In that paper, it is shown that large chains, large antichains, and diamond configurations embed into the Tukey types of p-points. However, this left open the question of what the exact structure of all Tukey types below a given p-point is. Recall that we use the terminology *initial Tukey structure* below an ultrafilter $\mathcal{U}$ to refer to the structure of the Tukey types of *all* nonprincipal ultrafilters Tukey reducible to $\mathcal{U}$ (including $\mathcal{U}$).

Todorcevic [24] showed that the initial Tukey structure below a Ramsey ultrafilter on ω consists exactly of one Tukey type, namely that of the Ramsey ultrafilter. Dobrinen and Todorcevic [11,12], showed that for each $1 \leq \alpha < \omega_1$, there are Ramsey spaces with associated ultrafilters which have initial Tukey and initial Rudin-Keisler structures which are decreasing chains of order type $\alpha + 1$. This left open the following questions from the Introduction, which we restate here.

Question 1 What are the possible initial Tukey structures for ultrafilters on a countable base set?

Question 2 What are the possible initial Rudin-Keisler structures for ultrafilters on a countable base set?

Question 3 For a given ultrafilter $\mathcal{U}$, what is the structure of the Rudin-Keisler ordering of the isomorphism classes of ultrafilters Tukey reducible to $\mathcal{U}$?

In this section, we answer Questions 1–3 for all Ramsey filters associated with a Ramsey space constructed from a generating sequence with Fraïssé classes which have the Order-Prescribed Free Amalgamation Property. The results in Theorems 60 and 67 show the surprising fact that the structure of the Fraïssé classes used for the generating sequence have bearing on the initial Rudin-Keisler structures, but not on the intial Tukey structures.

In this section we use topological Ramsey spaces constructed from generating sequences to identify some initial structures in the Tukey types of p-points. The next theorem is one of the main results, and will be proved at the end of this section.

Theorem 60 *Let $\mathcal{C}$ be a Ramsey filter on a Ramsey space constructed from a generating sequence for Fraïssé classes of ordered relational structures with the Ramsey property and the OPFAP.*

(1) If $J < \omega$, then the initial Tukey structure of all ultrafilters Tukey reducible to $\mathcal{C}$ is exactly $\mathcal{P}(J)$.
(2) If $J \le \omega$, then the Tukey ordering of the p-points Tukey reducible to $\mathcal{C}$ is isomorphic to the partial order $([J]^{<\omega}, \subseteq)$.

From Theorem 60, the following corollary is immediate.

Corollary 61 *It is consistent with ZFC that the following statements hold.*

(1) Every finite Boolean algebra appears as the initial Tukey structure below some p-point.
(2) The structure of the Tukey types of p-points contains the partial order $([\omega]^{<\omega}, \subseteq)$ as an initial structure.

The archetype for the proofs and results in this section comes from work in [11] showing that the initial Tukey structure below the ultrafilter associated with the space $\mathcal{R}_1$ is exactly a chain of length 2. (See Theorem 5.18 in [11] and results leading up to it.) The outline of that proof is now presented, as it will be followed in this section in more generality.

Outline of Proof of Theorem Theorem 5.18 in [11]. Recall that the space $\mathcal{R}_1$ in [11] is exactly the topological Ramsey space $\mathcal{R}(\langle \mathbf{A}_k : k < \omega \rangle)$ where $J = 1$ and for each $k < \omega$, $\mathbf{A}_{k,0}$ is a linear order of cardinality k. Let $\mathcal{C}$ be a maximal filter Ramsey for $\mathcal{R}_1$ and $\mathcal{U}_1$ be the ultrafilter it generates on the leaves of the base tree.

Theorem 5.18 in [11] was obtained in six main steps. (1) Theorem 20 from [10], every p-point has basic monotone reductions, was used to show that all ultrafilters

Tukey reducible to $\mathcal{U}_1$ are of the form $f(\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle)$ for some front on $\mathcal{C}$. (2) A localized version of the Ramsey-classifcation theorem for equivalence relations on fronts on $\mathcal{C}$ was shown to hold. (3) For each $n < \omega$, it was shown that the filter $\mathcal{Y}_{n+1}$ on the base set $\mathcal{R}_1(n)$ generated by $\mathcal{C} \upharpoonright \mathcal{R}_1(n)$ is a p-point ultrafilter. Furthermore, it was shown that $\mathcal{Y}_1 <_{RK} \mathcal{Y}_2 <_{RK} \dots$. (4) The localized Ramsey-classification theorem and the canonical equivalence relations were used to show that all ultrafilters Tukey reducible to $\mathcal{U}_1$ are isomorphic to an ultrafilter of $\mathcal{W}$-trees, where $\hat{\mathcal{S}} \backslash \mathcal{S}$ is a well-founded tree, $\mathcal{W} = (\mathcal{W}_s : s \in \hat{\mathcal{S}} \backslash \mathcal{S})$, and each $\mathcal{W}_s$ is isomorphic to $\mathcal{Y}_{n+1}$ for some $n < \omega$ or isomoprhic to $\mathcal{U}_0$. (5) The theory of uniform fronts was used to show that each ultrafilter generated by a $\mathcal{W}$-tree is isomorphic to a countable Fubini product from among the ultrafilters $\mathcal{Y}_n$, $n < \omega$. (6) The result on Fubini products was used to show that the Tukey structure of the non-principal ultrafilters on ω Tukey reducible to $\mathcal{U}_1$ is isomorphic to the two element Boolean algebra and that the p-points Tukey reducible to $\mathcal{U}_1$ are exactly $\{\mathcal{Y}_n : n < \omega\}$.

In order to avoid repeating phrases we fix some notation for the remainder of the section. Fix $1 \leq J \leq \omega$ and $\mathcal{K}_j$, $j \in J$, a collection of Fraïssé classes of finite ordered relational structures such that each $\mathcal{K}_j$ satisfies the Ramsey property and the OPFAP. Let $\mathcal{K}$ denote $(\mathcal{K}_j)_{j \in J}$. Let $\langle \mathbf{A}_k : k < \omega \rangle$ be a generating sequence, and let $\mathcal{R}$ denote the topological Ramsey space $\mathcal{R}(\langle \mathbf{A}_k : k < \omega \rangle)$.

Theorem 58 verifies that step (1) can be carried out for any Ramsey filter on $(\mathcal{R}, \leq)$. In the remainder of this section, we show that analogues of steps (2)–(6) can be carried out for any Ramsey filter on $(\mathcal{R}, \leq)$. The first part of step (2), proving the Ramsey-classifiication theorem for $\mathcal{R}$, was obtained in Theorem 38. We complete step (2) by showing that a localized version of Theorem 38 holds for Ramsey filters on $(\mathcal{R}, \leq)$. The analogue of step (3) is not as straightforward. First we introduce $\mathcal{K}_{\text{fin}}$ and then associate to each $\mathbf{B} \in \mathcal{K}_{\text{fin}}$ a p-point ultrafilter $\mathcal{U}_{\mathbf{B}}$ (see Notation 2). Then we show that the Rudin-Keisler structure of these p-points is isomorphic to $\tilde{\mathcal{K}}_{\text{fin}}$, the collection of equivalence classes of members of $\mathcal{K}_{\text{fin}}$, partially ordered by embeddability. Steps (4) and (5) are then generalized, the only difference being that the nodes of the $\mathcal{W}$-trees are taken to be the p-points $\mathcal{U}_{\mathbf{B}}$, $\mathbf{B} \in \mathcal{K}_{\text{fin}}$, from step (3). Step (6) will be completed at the end of the section by proving Theorem 60 and identifying initial structures in the Tukey types of ultrafilters.

The next theorem completes step (2) for topological Ramsey spaces constructed from generating sequences.

Theorem 62 *Let $\mathcal{C}$ be a Ramsey filter on a Ramsey space constructed from a generating sequence for Fraïssé classes of ordered relational structures with the Ramsey property and the OPFAP. If $\mathcal{C}$ is a Ramsey filter on $(\mathcal{R}, \leq)$, then for any front $\mathcal{F}$ on $\mathcal{R}$ and any equivalence relation R on $\mathcal{F}$, there exists a $C \in \mathcal{C}$ such that R is canonical on $\mathcal{F}|C$.*

Proof Since $\mathcal{C}$ satisfies the Abstract Nash-Williams Theorem for $\mathcal{R}$, and $\mathcal{C}$ is also selective for $\mathcal{R}$, by Lemma 3.8 in [18]. Thus, the proof of Theorem 38 can be relativized to $\mathcal{C}$. □

Next we complete step (3) for the general case by first identifying the p-points to be used as the nodes of the $\mathcal{W}$-trees we encounter in step (4), and then determining the Rudin-Keisler structure among these p-points.

Fact 63 *If $\mathcal{C} \subseteq \mathcal{R}$ is a Ramsey filter on $(\mathcal{R}, \leq)$, then for each $n < \omega$, $\mathcal{C} \upharpoonright \mathcal{R}(n) = \{\mathcal{R}(n)|C : C \in \mathcal{C}\}$ generates an ultrafilter on base set $\mathcal{R}(n)$.*

Notation 2 Suppose that $\mathcal{C} \subseteq \mathcal{R}$ is a Ramsey filter on $(\mathcal{R}, \leq)$. For each $n < \omega$, define $\mathcal{Y}_{n+1}$ to be the ultrafilter on $\mathcal{R}(n)$ generated by $\mathcal{C} \upharpoonright \mathcal{R}(n)$. Define $\mathcal{Y}_0 = \pi_{\text{depth}}(\mathcal{Y}_1)$ and $\mathcal{Y}_{\langle\rangle} = \pi_{\langle\rangle}(\mathcal{Y}_1)$. Let

$$\mathcal{K}_{\text{fin}} = \{(\mathbf{B}_j)_{j \in K} : K \in [J]^{<\omega} \text{ and } (\mathbf{B}_j)_{j \in K} \in (\mathcal{K}_j)_{j \in K}\}. \tag{21}$$

For $\mathbf{B} = (\mathbf{B}_j)_{j \in K}$ and $\mathbf{C} = (\mathbf{C}_j)_{j \in L}$ in $\mathcal{K}_{\text{fin}}$, define $\mathbf{B} \leq \mathbf{C}$ if and only if $K \subseteq L$ and for all $j \in K$, $\mathbf{B}_j \leq \mathbf{C}_j$. Let $\tilde{\mathcal{K}}_{\text{fin}}$ denote the collection of equivalence classes of members of $\mathcal{K}_{\text{fin}}$. Then $\tilde{\mathcal{K}}_{\text{fin}}$ is partially ordered by $\leq$.

For $\mathbf{B} = (\mathbf{B}_j)_{j \in K} \in \mathcal{K}_{\text{fin}}$ with $K \neq \emptyset$, define the following.

(1) Define $J_{\mathbf{B}}$ to be K and define

$$\mathcal{B}(\mathbf{B}) = \bigcup_{n<\omega} \left\{ \langle n, (\mathbf{C}_j)_{j \in J_{\mathbf{B}}} \rangle : \forall j \in J_{\mathbf{B}}, \mathbf{C}_j \in \binom{\mathbf{A}_{n,j}}{\mathbf{B}_j} \right\}.$$

(2) Applying the joint embedding property once for each $j \in J_{\mathbf{B}}$ and using the definition of generating sequence, there is an n such that for each $j \in K$, $\mathbf{B}_j \leq \mathbf{A}_{n,j}$. Define $n(\mathbf{B})$ to be the smallest natural number n such that for each $j \in J_{\mathbf{B}}$, $\mathbf{B}_j \leq \mathbf{A}_{n,j}$.
(3) Let $\mathbb{I}(\mathbf{B})$ denote the sequence $(I_j)_{j \in J_{n(\mathbf{B})}}$ such that $\pi_{\mathbb{I}(\mathbf{B})}(\mathbb{A}(n(\mathbf{B}))) = \langle n(\mathbf{B}), \mathbf{C} \rangle$, where $\mathbf{C}$ is the lexicographical-least element of $\binom{(\mathbf{A}_{n(\mathbf{B}),j})_{j \in J_{\mathbf{B}}}}{\mathbf{B}}$.
(4) We use the slightly more compact notation $\pi_{\mathbf{B}}$ to denote the map $\pi_{\mathbb{I}(\mathbf{B})}$.
(5) Let $\mathcal{U}_{\mathbf{B}}$ denote the ultrafilter $\pi_{\mathbf{B}}(\mathcal{Y}_{n(\mathbf{B})+1})$ on the base set $\mathcal{B}(\mathbf{B})$.

We let $\emptyset$ denote the sequence in $\mathcal{K}_{\text{fin}}$ with $K = \emptyset$.

The next proposition describes the configuration of the ultrafilters $\mathcal{U}_{\mathbf{B}}$ with $\mathbf{B} \in \mathcal{K}_{\text{fin}}$ and the projection ultrafilters $\pi_{\mathbb{I}}(\mathcal{Y}_i)$ with $i < \omega$ and $\pi_{\mathbb{I}}$ a projection map on $\mathcal{R}(i)$, with respect to the Rudin-Keisler ordering. For the remainder of the section, if $\pi_{\mathbb{I}}$ is a projection map on $\mathcal{R}(i)$ with $\mathbb{I} = (I_j)_{j \in J_i}$, then we let $J_{\mathbb{I}}$ denote the set $\{j \in J_i : I_j \neq \emptyset\}$. Recall that we write $\mathcal{U} \cong \mathcal{V}$ to denote that the two ultrafilters are Rudin-Keisler equivalent.

Proposition 64 *Suppose that $\mathcal{C} \subseteq \mathcal{R}$ is a Ramsey filter on $(\mathcal{R}, \leq)$.*

(1) $\mathcal{Y}_0$ is a Ramsey ultrafilter and $\mathcal{Y}_1$ is not a Ramsey ultrafilter.
(2) For each $n < \omega$, $\mathcal{Y}_{n+1} = \mathcal{U}_{\mathbf{A}_n}$.
(3) For each $m < \omega$ and each projection map $\pi_{\mathbb{I}}$ with domain $\mathcal{R}(m)$, there exists $\mathbf{B} \in \mathcal{K}_{\text{fin}}$ such that $\pi_{\mathbb{I}}(\mathcal{Y}_{m+1}) \cong \mathcal{U}_{\mathbf{B}}$.
(4) For each $\mathbf{B} \in \mathcal{K}_{\text{fin}}$, $\mathcal{U}_{\mathbf{B}}$ is a rapid p-point.
(5) For $\mathbf{B}$ and $\mathbf{C}$ in $\mathcal{K}_{\text{fin}}$, $\mathcal{U}_{\mathbf{B}} \leq_{RK} \mathcal{U}_{\mathbf{C}}$ if and only if $\mathbf{B} \leq \mathbf{C}$ in $\mathcal{K}_{\text{fin}}$.

Proof (1) $\mathcal{Y}_0 = \pi_{\text{depth}}(\mathcal{Y}_1)$ is a Ramsey ultrafilter since $\{\pi''_{\text{depth}}\mathcal{R}(0) \restriction C : C \in \mathcal{R}\}$ is identical to the Ellentuck space. $\mathcal{Y}_1$ is not Ramsey since the map π_{depth} is not one-to-one on any element of $\mathcal{Y}_1$. (2) For each $n < \omega$, $n(\mathbf{A}_n) = n$ and $\pi_{\mathbf{A}_n}$ is the identity map on $\mathcal{B}(\mathbf{A}_n)$. Thus, $\mathcal{U}_{\mathbf{A}_n} = \pi_{\mathbf{A}_n}(\mathcal{Y}_{n+1}) = \mathcal{Y}_{n+1}$.

(3) Suppose that $\pi_{\mathbb{I}}$ is a projection map with domain $\mathcal{R}(m)$. Let $\mathbf{B} = (\mathbf{B}_i)_{i\in J_{\mathbb{I}}}$ be the substructure of $(\mathbf{A}_{m,j})_{j\in J_{\mathbb{I}}}$ such that $\pi_{\mathbb{I}}(\mathbb{A}(m)) = \langle m, \mathbf{B}\rangle$. Let $n = n(\mathbf{B})$, and let $\mathbf{C} = (\mathbf{C}_i)_{i\in J_{\mathbb{I}}}$ be the substructure of $(\mathbf{A}_{n,j})_{j\in J_{\mathbb{I}}}$ such that $\pi_{\mathbf{B}}(\mathbb{A}(n)) = \langle n, \mathbf{C}\rangle$. We will show that $\mathcal{U}_{\mathbf{B}} = \pi_{\mathbf{B}}(\mathcal{Y}_{n+1}) \cong \pi_{\mathbb{I}}(\mathcal{Y}_{m+1})$.

Let $f : \mathbf{B} \to (\mathbf{A}_{m,j})_{j\in J_{\mathbb{I}}}$ be the embedding with range $\mathbf{B}$ and $g : \mathbf{B} \to (\mathbf{A}_{n,j})_{j\in J_{\mathbb{I}}}$ be the embedding with range $\mathbf{C}$. By the amalgamation property for $\mathcal{K}_j$, $j \in J_{\mathbb{I}}$, and the definition of generating sequence, there exist $k < \omega$ and embeddings $r : (\mathbf{A}_{n,j})_{j\in J_{\mathbb{I}}} \to (\mathbf{A}_{k,j})_{j\in J_{\mathbb{I}}}$ and $s : (\mathbf{A}_{m,j})_{j\in J_{\mathbb{I}}} \to (\mathbf{A}_{k,j})_{j\in J_{\mathbb{I}}}$ such that $r \circ f = s \circ g$. Let $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ denote the substructures of $(\mathbf{A}_{k,j})_{j\in J_{\mathbb{I}}}$ generated by the ranges of $r \circ f$, s, and r, respectively. Let $\pi_{\mathbb{M}}$, $\pi_{\mathbb{N}}$ and $\pi_{\mathbb{F}}$ denote projection maps on $\mathcal{R}(k)$ such that $J_{\mathbb{M}} = J_{\mathbb{N}} = J_{\mathbb{F}} = J_{\mathbb{I}}$, $\pi_{\mathbb{M}}(\mathbb{A}(k)) = \langle k, \mathbf{G}\rangle$, $\pi_{\mathbb{N}}(\mathbb{A}(k)) = \langle k, \mathbf{H}\rangle$ and $\pi_{\mathbb{F}}(\mathbb{A}(k)) = \langle k, \mathbf{F}\rangle$. Since $r \circ f = s \circ g$, it follows that for all $y \in \mathcal{AR}_{k+1}$, $\pi_{\mathbf{B}} \circ \pi_{\mathbb{N}}(y(k)) = \pi_{\mathbb{I}} \circ \pi_{\mathbb{M}}(y(k)) = \pi_{\mathbb{F}}(y(k))$.

Let $X \in \mathcal{C}$ and consider the set $\mathcal{G} = \{x \in \mathcal{AR}_{n+1} : \exists y \in \mathcal{R}(k)|X,\ \pi_{\mathbb{F}}(y) = \pi_{\mathbf{B}}(x(n))\}$. Since $\mathcal{C}$ satisfies the Abstract Nash-Williams Theorem it follows that there exists a $Y \le X$ in $\mathcal{C}$ such that either $\mathcal{G} \cap \mathcal{AR}_{n+1}|Y = \emptyset$ or $\mathcal{AR}_{n+1}|Y \subseteq \mathcal{G}$. Since there exists $z \in \mathcal{AR}_{n+1}|Y$ such that $\pi_{\mathbb{F}}(Y(k)) = \pi_{\mathbf{B}}(z(n))$ it must be the case that $\mathcal{AR}_{n+1}|Y \subseteq \mathcal{G}$. By Fact 51 it follows that $\pi_{\mathbb{F}}(\mathcal{Y}_k) \cong \pi_{\mathbf{B}}(\mathcal{Y}_n)$. By a similar argument, we also have $\pi_{\mathbb{F}}(\mathcal{Y}_k) \cong \pi_{\mathbb{I}}(\mathcal{Y}_m)$. Thus, $\mathcal{U}_{\mathbf{B}} = \pi_{\mathbf{B}}(\mathcal{Y}_n) \cong \pi_{\mathbb{I}}(\mathcal{Y}_m)$.

(4) Let K be a finite subset of J and $\mathbf{B} = (\mathbf{B}_j)_{j\in K} \in \mathcal{K}_{\text{fin}}$. Suppose that $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ is a sequence of sets in $\mathcal{U}_{\mathbf{B}}$. Then there exists a sequence $C_0 \ge C_1 \ge C_2 \ge \cdots$ of elements of $\mathcal{C}$ such that for each $i < \omega$, $\pi''_{\mathbf{B}}(\mathcal{R}(n(\mathbf{B})) \restriction C_i) \subseteq X_i$. Since every Ramsey filter on $(\mathcal{R}, \le)$ is also a selective filter for $(\mathcal{R}, \le)$, there exists $C \in \mathcal{C}$ such that for each $i < \omega$, $C/r_i(C) \le C_i$. Since each $\mathcal{K}_j$, $j \in K$, consists of finite structures and K is finite, it follows that for each $i < \omega$, $\pi''_{\mathbf{B}}(\mathcal{R}(n) \restriction C) \subseteq^* \pi''_{\mathbf{B}}(\mathcal{R}(n) \restriction C_i)$. Therefore $\mathcal{U}_{\mathbf{B}}$ is a p-point.

Let $h : \omega \to \omega$ be a strictly increasing function. Linearly order $\mathcal{B}(\mathbf{B})$ so that $\langle i, \mathbf{C}\rangle$ comes before $\langle j, \mathbf{D}\rangle$ whenever $i < j$. For each $B \in \mathcal{R}$, there is a $C \le B$ such that $\pi_{\text{depth}}(C(n-1)) > h(1)$, $\pi_{\text{depth}}(C(n(\mathbf{B}))) > h(1 + |\mathcal{B}(\mathbf{B}) \restriction \langle n(\mathbf{B}), \mathbf{A}_{n(\mathbf{B})}\rangle|)$, and in general, for $k > n(\mathbf{B})$,

$$\pi_{\text{depth}}(C(k)) > h\left(\sum_{i=n}^{k} |\mathcal{B}(\mathbf{B}) \restriction \langle i, \mathbf{A}_i\rangle|\right). \tag{22}$$

Since $\mathcal{C}$ is selective for $\mathcal{R}$, there is a $C \in \mathcal{C}$ with this property, which yields that $\mathcal{U}_{\mathbf{B}}$ is rapid.

(5) ($\Leftarrow$) Suppose that $\mathbf{B} = (\mathbf{B}_j)_{j\in K}$ and $\mathbf{C} = (\mathbf{C}_j)_{j\in L}$ are elements of $\mathcal{K}_{\text{fin}}$ and $\mathbf{B} \le \mathbf{C}$. Let $\mathbb{I}(\mathbf{C}) = (I_j)_{j\in J_{n(\mathbf{C})}}$. Then $K \subseteq L$ and there is a sequence $\mathbb{I} = (I'_j)_{j\in J_{n(\mathbf{C})}}$ such that for each $j \in J_{n(\mathbf{C})}$, $I'_j \subseteq I_j$, and the structure $\mathbf{D}$ in $\mathcal{K}_{\text{fin}}$ such that $\pi_{\mathbb{I}}(\mathbb{A}(n(\mathbf{C}))) = \langle n(\mathbf{C}), \mathbf{D}\rangle$ is isomorphic to $\mathbf{B}$. By the work in part (3) of this proposi-

tion, $\pi_{\mathbb{I}}(\mathcal{Y}_{n(\mathbf{C})+1}) \cong \pi_{\mathbf{B}}(\mathcal{Y}_{n(\mathbf{B})+1}) = \mathcal{U}_{\mathbf{B}}$. Since for each $j \in J_{n(\mathbf{C})}$, $I'_j \subseteq I_j$, we have $\pi_{\mathbb{I}}(\mathcal{Y}_{n(\mathbf{C})+1}) \leq_{RK} \pi_{\mathbf{C}}(\mathcal{Y}_{n(\mathbf{C})+1}) = \mathcal{U}_{\mathbf{C}}$. Hence, $\mathcal{U}_{\mathbf{B}} \leq_{RK} \mathcal{U}_{\mathbf{C}}$.

(5) ($\Rightarrow$) Next suppose that $(\mathbf{B}_j)_{j\in K}$ and $(\mathbf{C}_j)_{j\in L}$ are elements of $\mathcal{K}_{\text{fin}}$ and $\mathcal{U}_{\mathbf{B}} \leq_{RK} \mathcal{U}_{\mathbf{C}}$.

Lemma 65 *For each nonprincipal ultrafilter $\mathcal{V}$ on ω with $\mathcal{V} \leq_{RK} \mathcal{U}_{\mathbf{C}}$, there exists $\mathbf{D} \in \mathcal{K}_{\text{fin}}$ such that $\mathbf{D} \leq \mathbf{C}$ and $\mathcal{V} \cong \mathcal{U}_{\mathbf{D}}$. Furthermore, if $\mathcal{V} \cong \mathcal{U}_{\mathbf{C}}$ then $J_{\mathbf{D}} = J_{\mathbf{C}}$ and for all $j \in J_{\mathbf{C}}$, $\mathbf{C}_j \cong \mathbf{D}_j$.*

Proof Suppose that $\mathcal{V}$ is a nonprincipal ultrafilter on ω such that $\mathcal{V} \leq_{RK} \mathcal{U}_{\mathbf{C}}$. Then there is a function $\theta : \mathcal{B}(\mathbf{C}) \to \omega$ such that $\theta(\mathcal{U}_{\mathbf{C}}) = \mathcal{V}$. Since $\theta \circ \pi_{\mathbf{C}} : \mathcal{R}(n(\mathbf{C})) \to \omega$, Theorem 62 implies that there exist an $X \in \mathcal{C}$ and a projection map $\pi_{\mathbb{I}}$ on $\mathcal{R}(n(\mathbf{C}))$ such that for all $y, z \in \mathcal{R}(n(\mathbf{C})) \restriction X$,

$$\theta \circ \pi_{\mathbf{C}}(y) = \theta \circ \pi_{\mathbf{C}}(z) \text{ if and only if } \pi_{\mathbb{I}}(y) = \pi_{\mathbb{I}}(z). \tag{23}$$

Suppose $\mathbb{I} = (I_j)_{j\in J_{n(\mathbf{C})}}$ and $\mathbb{I}(\mathbf{C}) = (I'_j)_{j\in J_{n(\mathbf{C})}}$. Let $\mathbf{D} \in \mathcal{K}_{\text{fin}}$ such that $\pi_{\mathbb{I}}(\mathbb{A}(n(\mathbf{C}))) = \langle n(\mathbf{C}), \mathbf{D}\rangle$. If there exists $j \in J_{n(\mathbf{C})}$ such that $I'_j \nsubseteq I_j$ or there exists $j \in J_{\mathbb{I}}$ such that $\mathbf{D}_j \not\leq \mathbf{C}_j$, then there exist $s, t \in \mathcal{R}(n(\mathbf{C})) \restriction X$ such that $\pi_{\mathbb{I}}(s) \neq \pi_{\mathbb{I}}(t)$ and $\pi_{\mathbf{C}}(s) = \pi_{\mathbf{C}}(t)$. However, this is a contradiction to Eq. (23). Therefore, $J_{\mathbf{D}} \subseteq J_{\mathbf{C}}$ and for all $j \in J_{n(\mathbf{D})}$, $\mathbf{D}_j \leq \mathbf{C}_j$, *i.e.* $\mathbf{D} \leq \mathbf{C}$. Additionally, Eq. (23) shows that $\mathcal{U}_{\mathbf{D}} \cong \theta(\mathcal{U}_{\mathbf{C}}) = \mathcal{V}$.

Next suppose that $\mathcal{V} \cong \mathcal{U}_{\mathbf{C}}$. Then there exists $Y \in \mathcal{C}$ such that $Y \leq X$ and θ is injective on $\pi''_{\mathbf{C}}(\mathcal{R}(n(\mathbf{C})) \restriction Y)$. If there is a $j \in J_{n(\mathbf{C})}$ such that $I_j \nsubseteq I'_j$ or there is a $j \in J_{\mathbb{I}}$ such that $\mathbf{C}_j \not\leq \mathbf{D}_j$, then there are $s, t \in \mathcal{R}(n(\mathbf{C})) \restriction Y$ such that $\pi_{\mathbb{I}}(s) = \pi_{\mathbb{I}}(t)$ and $\pi_{\mathbf{C}}(s) \neq \pi_{\mathbf{C}}(t)$. However, this contradicts the fact that θ is injective on $\pi''_{\mathbf{C}}(\mathcal{R}(n(\mathbf{C})) \restriction Y)$. Therefore, $J_{\mathbf{C}} \subseteq J_{\mathbf{D}}$ and for all $j \in J_{n(\mathbf{C})}$, $\mathbf{C}_j \leq \mathbf{D}_j$, that is, $\mathbf{C} \leq \mathbf{D}$. Thus, $J_{\mathbf{D}} = J_{\mathbf{C}}$ and for all $j \in J_{\mathbf{C}}$, $\mathbf{C}_j \cong \mathbf{D}_j$. □

Since $\mathcal{U}_{\mathbf{B}} \leq_{RK} \mathcal{U}_{\mathbf{C}}$, Lemma 65 shows that $\mathbf{B} \leq \mathbf{C}$. □

In what follows, we omit any proofs of results which follow the exact same argument as their counterparts in the proof of Theorem 5.10 in [11]. The following makes use of the correspondence between iterated Fubini products of ultrafilters and so-called ultrafilters of $\boldsymbol{\mathcal{W}}$-trees on a flat-top front, (see Definition 3.2 and Facts 3.4 and 3.4 in [6]). A uniform front is, in particular, a flat-top front, and the projection of the uniform front $\mathcal{C}|C$ in the next theorem to $\hat{\mathcal{S}}$ will also be a flat-top front.

Theorem 66 *Suppose that $\mathcal{C}$ is a Ramsey filter on $(\mathcal{R}, \leq)$. If $\mathcal{V}$ is a non-principal ultrafilter and $\mathcal{C} \geq_T \mathcal{V}$, then $\mathcal{V}$ is isomorphic to a Fubini iterate of p-points from among $\mathcal{U}_{\mathbf{B}}$, $\mathbf{B} \in \mathcal{K}_{\text{fin}}$. Precisely, $\mathcal{V}$ is isomorphic to an ultrafilter of $\boldsymbol{\mathcal{W}}$-trees, where $\hat{\mathcal{S}}\backslash\mathcal{S}$ is a well-founded tree, $\boldsymbol{\mathcal{W}} = (\mathcal{W}_s : s \in \hat{\mathcal{S}}\backslash\mathcal{S})$, and each $\mathcal{W}_s$ is isomorphic to $\mathcal{U}_{\mathbf{B}}$, for some $\mathbf{B} \in \mathcal{K}_{\text{fin}}$.*

Proof Suppose that $\mathcal{C}$ and $\mathcal{V}$ are given and satisfy the assumptions of the theorem. By Proposition 50 and Lemma 62 there are a front $\mathcal{F}$ on $\mathcal{C}$, a function $f : \mathcal{F} \to \omega$, and a $C \in \mathcal{C}$ such that the following hold:

(1) The equivalence relation induced by f on $\mathcal{F}|C$ is canonical.
(2) $\mathcal{V} = f(\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle)$.

A straightforward induction argument on the rank of fronts, along with the fact that $\mathcal{F}$ is Ramsey, shows that there is a $C' \le C \in \mathcal{C}$ such that $\mathcal{F}|C'$ is a uniform front on $\mathcal{C}|C'$. From now on, we will abuse notation and let $\mathcal{F}$ denote $\mathcal{F}|C'$ and $\mathcal{C}$ denote $\mathcal{C}|C'$.

Let $\mathcal{S} = \{\varphi(t) : t \in \mathcal{F}\}$, where φ is the inner Nash-Williams map from Theorem 38 which represents the canonical equivalence relation. The filter $\mathcal{W}$ on the base set $\mathcal{S}$ generated by $\varphi(\mathcal{C} \upharpoonright \mathcal{F})$ is an ultrafilter, and $\mathcal{W} \cong \mathcal{V}$. We omit the proof of this fact, since it follows from exactly the same argument as its counterpart in the proof of Theorem 5.10 in [11].

Recall from the proof of Theorem 38 that for each $t \in \mathcal{F}$ and $i < |t|$, there is a projection map $\pi_{r_i(t)}$ defined on $\mathcal{R}(i)$ such that $\varphi(t) = \bigcup_{i<|t|} \pi_{r_i(t)}(t(i))$. We now extend φ to a map on all of $\hat{\mathcal{F}}$ by defining $\varphi(r_j(t)) = \bigcup_{i<j} \pi_{r_i(t)}(t(i))$, for $t \in \mathcal{F}$ and $j \le |t|$.

Let $\hat{\mathcal{S}}$ denote the collection of all initial segments of elements of $\mathcal{S}$. Thus, $\hat{\mathcal{S}} = \{\varphi(w) : w \in \hat{\mathcal{F}}\}$. $\hat{\mathcal{S}}$ forms a well-founded tree under the ordering $\sqsubseteq$. For $s \in \hat{\mathcal{S}} \setminus \mathcal{S}$, define $\mathcal{W}_s$ to be the filter on the base set $\{\pi_{r_j(t)}(u) : u \in \mathcal{R}(j)\}$ generated by the sets $\{\pi_{r_j(t)}(u) : u \in \mathcal{R}(j)|X/r_j(t)\}$, $X \in \mathcal{C}$, for any (all) $t \in \mathcal{F}$ such that $s \sqsubseteq \varphi(t)$ and $j < |t|$ maximal such that $\varphi(r_j(t)) = s$. The proof of the next claim follows exactly as in [11].

Claim 17 *For each $s \in \hat{\mathcal{S}} \setminus \mathcal{S}$, $\mathcal{W}_s$ is an ultrafilter which is generated by the collection of $\{\pi_{r_j(t)}(u) : u \in \mathcal{R}(j)|X\}$, $X \in \mathcal{C}$, for any $t \in \mathcal{F}$ and $j < |t|$ maximal such that $\varphi(r_j(t)) = s$.*

The proof of the next claim is included, as it differs from its counterpart in the proof of Theorem 5.1 in [11].

Claim 18 *Let $s \in \hat{\mathcal{S}} \setminus \mathcal{S}$. Then $\mathcal{W}_s$ is isomorphic to $\mathcal{U}_{\mathbf{B}}$ for some $\mathbf{B} \in \mathcal{K}_{\mathrm{fin}}$.*

Proof Fix $t \in \mathcal{F}$ and $j < |t|$ with j maximal such that $\varphi(r_j(t)) = s$. Suppose that $\pi_{r_j(t)} = \pi_{\mathrm{depth}}$. Then for each $X \in \mathcal{C}$, $\{\pi_{r_j(t)}(u) : u \in \mathcal{R}(j)|X\} = \pi_{\mathrm{depth}}(\mathcal{R}(j)|X) \in \mathcal{Y}_0$. Since $\mathcal{W}_s$ is non-principal, $\mathcal{W}_s = \mathcal{Y}_0 = \mathcal{U}_{\emptyset}$, by Fact 51. If $\pi_{r_j(t)} = \pi_{\mathbb{I}}$, then for each $X \in \mathcal{C}$, $\{\pi_{r_j(t)}(u) : u \in \mathcal{R}(j)|X/t\} \subseteq \{\pi_{\mathbb{I}}(u) : u \in \mathcal{R}(j)|X\} \in \pi_{\mathbb{I}}(\mathcal{Y}_{j+1})$. Thus, by Fact 51, $\mathcal{W}_s = \pi_{\mathbb{I}}(\mathcal{Y}_{j+1})$. By Proposition 64 (3), there exists $\mathbf{B} \in \mathcal{K}_{\mathrm{fin}}$ such that $\mathcal{W}_s \cong \mathcal{U}_{\mathbf{B}}$. □

The proof of the next claim follows as in [11].

Claim 19 *$\mathcal{W}$ is the ultrafilter generated by $\boldsymbol{\mathcal{W}}$-trees, where $\boldsymbol{\mathcal{W}} = (\mathcal{W}_s : s \in \hat{\mathcal{S}} \setminus \mathcal{S})$.*

The previous claims show that $\mathcal{V}$ is isomorphic to the ultrafilter $\mathcal{W}$ on the base $\mathcal{S}$ generated by the $\boldsymbol{\mathcal{W}}$-trees, where for each $s \in \hat{\mathcal{S}} \setminus \mathcal{S}$, $\mathcal{W}_s$ is isomorphic to $\mathcal{U}_{\mathbf{B}}$ for some $\mathbf{B} \in \mathcal{K}_{\mathrm{fin}}$. By the correspondence of ultrafilters of $\boldsymbol{\mathcal{W}}$-trees on $\mathcal{S}$ and iterated Fubini products, we conclude that $\mathcal{V}$ is isomorphic to a Fubini iterate of p-points from among $\mathcal{U}_{\mathbf{B}}$, $\mathbf{B} \in \mathcal{K}_{\mathrm{fin}}$. □

Theorem 67 *Suppose that $\mathcal{C}$ is a Ramsey filter on $(\mathcal{R}, \le)$. The Rudin-Keisler ordering of the p-points Tukey reducible to $\mathcal{C}$ is isomorphic to the partial order $(\tilde{\mathcal{K}}_{\mathrm{fin}}, \le)$. In particular, if $|J| < \omega$, then the initial Rudin-Keisler structure below $\mathcal{C}$ is isomorphic to the partial order $(\tilde{\mathcal{K}}_{\mathrm{fin}}, \le)$.*

Proof The first statement follows from Theorem 66, the correspondence between and iterated Fubini products of ultrafilters (see Facts 3.4 and 3.4 in [6]), and the fact that a Fubini product of ultrafilters is never a p-point. If $|J| < \omega$, then $\{\mathcal{AR}_1|X : X \in \mathcal{C}\}$ generates a p-point on the base set $\mathcal{AR}_1$, and in this case, every ultrafilter Rudin-Keisler reducible to $\mathcal{C}$ is a p-point. □

Proposition 68 *Suppose that* $\mathbf{B}$ *and* $\mathbf{C}$ *are in* $\mathcal{K}_{\mathrm{fin}}$. *Then* $\mathcal{U}_{\mathbf{B}} \le_T \mathcal{U}_{\mathbf{C}}$ *if and only if* $J_{\mathbf{B}} \subseteq J_{\mathbf{C}}$. *Hence,* $\mathcal{U}_{\mathbf{B}} \equiv_T \mathcal{U}_{\mathbf{C}}$ *if and only if* $J_{\mathbf{B}} = J_{\mathbf{C}}$.

Proof Assume $J_{\mathbf{B}} \subseteq J_{\mathbf{C}}$. By Proposition 64, $\mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}} \le_T \mathcal{U}_{\mathbf{B}}$, since $(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}} \le \mathbf{B}$. Define $g : \mathcal{C} \restriction \mathcal{B}((\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}) \to \mathcal{C} \restriction \mathcal{B}(\mathbf{B})$ by $g(\mathcal{B}((\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}) \restriction X) = \mathcal{B}(\mathbf{B}) \restriction X$. g is well-defined on a cofinal subset of $\mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}}$, since from the set $\mathcal{B}((\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}) \restriction X$ one can reconstruct $\{\langle k_n, (\mathbf{X}_{n,j})_{j\in J_{\mathbf{B}}}\rangle : n < \omega\}$. Since g is a monotone cofinal map from a cofinal subset of $\mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}}$ into a cofinal subset of $\mathcal{U}_{\mathbf{B}}$, we have $\mathcal{U}_{\mathbf{B}} \le_T \mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}}$. Hence $\mathcal{U}_{\mathbf{B}} \equiv_T \mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}}$. By a similar argument, $\mathcal{U}_{\mathbf{C}} \equiv_T \mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{C}}}}$. Since $J_{\mathbf{B}} \subseteq J_{\mathbf{C}}$, Proposition 64 implies that $\mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}} \le_T \mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{C}}}}$. Therefore $\mathcal{U}_{\mathbf{B}} \le_T \mathcal{U}_{\mathbf{C}}$.

Now suppose that $J_{\mathbf{B}} \not\subseteq J_{\mathbf{C}}$. Since $J_{\mathbf{C}}$ is finite, the p-point ultrafilter $\mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{C}}}}$ is Tukey equivalent to a Ramsey filter on $(\mathcal{R}(\langle(\mathbf{A}_{k,j})_{j\in J_{\mathbf{C}}} : k < \omega\rangle), \le)$. By Theorem 66, if $\mathcal{V}$ is a p-point and $\mathcal{V} \le_T \mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{C}}}}$ then $\mathcal{V}$ is isomorphic to some $\mathcal{U}_{\mathbf{D}}$ for some $\mathbf{D} \in \bigcup_{L\subseteq J_{\mathbf{C}}} (\mathcal{K}_j)_{j\in L}$ or isomorphic to $\mathcal{Y}_0$. By Proposition 64 (5), $\mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}} \not\le_{RK} \mathcal{U}_{\mathbf{D}}$ for each $\mathbf{D} \in \bigcup_{J'\subseteq J_{\mathbf{C}}} (\mathcal{K}_j)_{j\in J'}$, and also $\mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}} \not\le_{RK} \mathcal{Y}_0$. So $\mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{B}}}} \not\le_T \mathcal{U}_{(\mathbf{A}_{0,j})_{j\in J_{\mathbf{C}}}}$. Therefore $\mathcal{U}_{\mathbf{B}} \not\le_T \mathcal{U}_{\mathbf{C}}$. □

Final Argument for Proof of Theorem 60. For now, let J be either finite or ω. We prove (2) first. Let $\mathcal{V}$ be any p-point Tukey reducible to $\mathcal{C}$. Since a Fubini product of ultrafilters is never a p-point, it follows from Theorem 67 that $\mathcal{V}$ is isomorphic to $\mathcal{U}_{\mathbf{B}}$ for some $\mathbf{B} \in \mathcal{K}_{\mathrm{fin}}$. In particular, $\mathcal{V}$ is Tukey equivalent to $\mathcal{U}_{\mathbf{B}}$. Proposition 68 shows that the Tukey type of $\mathcal{U}_{\mathbf{B}}$ is completely determined by the index set $J_{\boldsymbol{B}}$. Therefore, (2) holds.

Now suppose J is finite. By Theorem 67, each ultrafilter $\mathcal{V}$ Tukey reducible to $\mathcal{C}$ is isomorphic to a Fubini iterate of ultrafilters from among $\mathcal{U}_{\mathbf{B}}$, $\mathbf{B} \in \mathcal{K}_{\mathrm{fin}}$. Since J is finite, it follows from Proposition 68 that there are only finitely many Tukey types of p-points Tukey reducible to $\mathcal{C}$. By Corollary 37 in [10], for each $\mathcal{U}_{\mathbf{B}}$, its Tukey type is the same as the Tukey type of any Fubini power of itself, since $\mathcal{U}_{\mathbf{B}}$ is a rapid p-point. Thus, each Fubini iterate from among the $\mathcal{U}_{\mathbf{B}}$, $\mathbf{B} \in \mathcal{K}_{\mathrm{fin}}$, has Tukey type equal to some such $\mathcal{U}_{\mathbf{B}}$. Therefore, the initial Tukey structure below $\mathcal{C}$ is exactly $\mathcal{P}(J)$. □

We finish by pointing out the implications the theorems in this section have for the specific examples in Sect. 4.

Example 69 (n-arrow, not $(n+1)$-arrow ultrafilters) Let $J = 1$ and fix $n \ge 2$. Recall that the space $\mathcal{A}_n$ is defined to be the space $\mathcal{R}(\langle \mathbf{A}_k : k < \omega\rangle)$, where $\langle \mathbf{A}_k : k < \omega\rangle$ is some generating sequence for $\mathcal{K}_0$, the class of all finite $(n+1)$-clique-free ordered graphs.

Suppose $\mathcal{C}$ is a Ramsey filter on $(\mathcal{A}_n, \leq)$. Then $\mathcal{U}_{\mathcal{A}_n}$, defined to be the filter on base set $\mathcal{R}(0)$ generated by the sets $\mathcal{R}(0)|C$, $C \in \mathcal{C}$, is a p-point ultrafilter which is Tukey equivalent to $\mathcal{C}$. By Theorem 67, the Rudin-Keisler structure of the p-points Tukey reducible to $\mathcal{U}_{\mathcal{A}_n}$ is isomorphic to the collection of all equivalence classes of members of $\mathcal{K}_0$, partially ordered by embedability. By Theorem 60, the initial Tukey structure below $\mathcal{U}_{\mathcal{A}_n}$ is exactly a chain of length 2.

Example 70 (Hypercube spaces) Let $J \leq \omega$ and for each $j \in J$, let $\mathcal{K}_j$ be the class of finite linear orders. Let $\langle \mathbf{A}_k : k < \omega \rangle$ be a generating sequence such that for each $k < \omega$ and each $j \in J_k$, $\mathbf{A}_{k,j}$ is a k-element linear order. Recall that the space $\mathcal{H}^J$ is defined to be the space $\mathcal{R}(\langle \mathbf{A}_k : k < \omega \rangle)$.

If $\mathcal{C}$ is a Ramsey filter on $(\mathcal{H}^J, \leq)$ then by Theorem 67, the Rudin-Keisler structure of the p-points Tukey reducible to $\mathcal{C}$ is isomorphic to the partial order $(\tilde{\mathcal{K}}_{\mathrm{fin}}, \leq)$.

If $J < \omega$ then $(\tilde{\mathcal{K}}_{\mathrm{fin}}, \leq)$ is isomorphic to the partial order $(\omega^J, \leq)$ via the map sending $\mathbf{B} \mapsto (\|\mathbf{B}_0\|, \|\mathbf{B}_1\|, \ldots, \|\mathbf{B}_{J-1}\|)$. (If $j \notin J_{\mathbf{B}}$, then we assume $\|\mathbf{B}_j\| = 0$.) Moreover, the initial Tukey structure below $\mathcal{C}$ is exactly $\mathcal{P}(J)$.

Let $C_0(\omega)$ denote the collection of sequences of natural numbers which are eventually zero. (For $(x_i)_{i<\omega}$ and $(y_i)_{i<\omega}$ in $C_0(\omega)$, $(x_i)_{i<\omega} \leq (y_i)_{i<\omega}$ iff for all $i < \omega$, $x_i \leq y_i$.) If $J = \omega$, then $(\tilde{\mathcal{K}}_{\mathrm{fin}}, \leq)$ is isomorphic to the partial order $(C_0(\omega), \leq)$ via the map $\mathbf{B} \mapsto (\|\mathbf{B}_0\|, \|\mathbf{B}_1\|, \ldots, \|\mathbf{B}_{J_{\mathbf{B}}}\|, 0, 0, 0, \ldots)$. Further, the structure of the Tukey types of the p-points Tukey reducible to $\mathcal{C}$ is exactly $[\omega]^{<\omega}$.

These examples are just prototypes of what can be achieved by topological Ramsey spaces constructed from generating sequences. Based on the work in this paper, many examples of topological Ramsey spaces can be constructed, with associated ultrafilters having a wide range of partition properties and initial Rudin-Keisler and Tukey structures.

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