Applications of High Dimensional Ellentuck spaces

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Boolean algebras
Lattices
Algebraic logic, universal Algebra
Set theory
Topology - general, point-free, set-theoretic

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The **Ellentuck space** is the space $[\omega]^\omega$ with topology generated by basic open sets

$$[s, A] = \{ X \in [\omega]^\omega : s \subseteq X \subseteq A \},$$

where $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$.

**Thm.** (Ellentuck) The Ellentuck space is a topological Ramsey space: Given $\mathcal{X} \subseteq [\omega]^\omega$ with the property of Baire, for any basic open set $[s, A]$, there is a member $B \in [s, A]$ such that

$$\text{either } [s, B] \subseteq \mathcal{X} \text{ or else } [s, B] \cap \mathcal{X} = \emptyset.$$ 

Forcing with members of the Ellentuck space partially ordered by $\subseteq^*$ adds a Ramsey ultrafilter.
Forcing Ultrafilters

$([\omega]^\omega, \subseteq^*)$ is forcing equivalent to $\mathcal{P}(\omega)/\text{Fin}$.

A natural extension of this Boolean algebra: $\mathcal{P}(\omega \times \omega)/\text{Fin} \otimes \text{Fin}$.

$X \in \text{Fin} \otimes \text{Fin}$ iff $X \subseteq \omega \times \omega$ and $\forall \infty n \in \omega$, $\{i \in \omega : (n, i) \in X\} \in \text{Fin}$.

$\mathcal{P}(\omega^2)/\text{Fin}^\otimes^2$ adds an ultrafilter $\mathcal{U}_2$, the next best thing to a p-point:

$\mathcal{U}_2 \rightarrow (\mathcal{U}_2)_{r,4}^2$.

The projection to the first coordinates, $\pi_1(\mathcal{U}_2)$, is a Ramsey ultrafilter, generic for $\pi_1(\mathcal{P}(\omega^2)/\text{Fin}^\otimes^2) \cong \mathcal{P}(\omega)/\text{Fin}$. 
Extending $\text{Fin} \otimes^2$ to all uniform barriers

Recursively construct ideals on $\omega^{k+1}$: $\text{Fin} \otimes^{k+1} = \text{Fin} \otimes \text{Fin} \otimes^k$.

$\mathcal{P}(\omega^k) / \text{Fin} \otimes^k$ forces an ultrafilter $\mathcal{U}_k$: for each $j < k$, $\pi_j(\mathcal{U}_k) \cong \mathcal{U}_j$.

Replace $\omega^k$ by $[\omega]^k$; $\text{Fin} \otimes^k$ by the ideal $I_k$ on $[\omega]^k$ determined by $\text{Fin} \otimes^k$.

$\mathcal{P}([\omega]^k) / I_k \cong \mathcal{P}(\omega^k) / \text{Fin} \otimes^k$.

$[\omega]^k$ is a uniform barrier on $\omega$ of rank $k$.

This construction of $I_k$ can be extended to all uniform barriers on $\omega$. 
Example. Schreier barrier: \( S = \{ s \in [\omega]^\omega : |s| = \text{min } s + 1 \} \).

For \( X \subseteq S \), \( X_n = \{ s \in X : \text{min } s = n \} \).

\[ I_S = \{ X \subseteq S : \forall \omega \ (X_n \in I_S) \}. \]

For any uniform barrier \( B \) on \( \omega \), \( \mathcal{P}(B)/I_B \) forces an ultrafilter \( \mathcal{U}_B \) on countable base set \( B \).

**Fact.** If \( B \) projects to \( C \), then \( \mathcal{P}(C)/I_C \) embeds as a complete subalgebra of \( \mathcal{P}(B)/I_B \), and \( \mathcal{U}_C \) is isomorphic to a projection of \( \mathcal{U}_B \).
A function $f : \mathcal{U} \rightarrow \mathcal{V}$ between ultrafilters is **cofinal** if $f$ maps each filter base for $\mathcal{U}$ to a filter base for $\mathcal{V}$.

$\mathcal{U}$ is **Tukey reducible** to $\mathcal{U} \geq_T \mathcal{V}$ iff there is a cofinal map from $\mathcal{U}$ into $\mathcal{V}$.

The equivalence relation defined by $\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$ is a coarsening of the Rudin-Keisler equivalence relation of isomorphism.

**Thm.**

1. (Folklore) The ultrafilter $\mathcal{U}_2$ forced by $\mathcal{P}(\omega^2)/\text{Fin} \otimes^2$ is Rudin-Keisler minimal above the Ramsey ultrafilter $\pi_1(\mathcal{U}_2)$.

2. (Blass, D., Raghavan) $\mathcal{U}_2 \geq_T \pi_1(\mathcal{U}_2)$ and $\mathcal{U}_2$ is not Tukey maximal.
Initial Tukey Structures

So what exactly is Tukey below $\mathcal{U}_2$?

**Thm.** [D1]
1. $\mathcal{U}_2$ is Tukey minimal above its projected Ramsey ultrafilter $\pi_1(\mathcal{U}_2)$.
2. For each $k \geq 2$, the ultrafilter $\mathcal{U}_k$ forced by $\mathcal{P}(\omega^k)/\text{Fin}^\otimes k$ has initial Tukey structure exactly a chain of length $k$. Likewise for its initial Rudin-Keisler structure.
3. [D2 and unpublished] For each uniform barrier $B$ of infinite rank, $\mathcal{U}_B$ has initial Tukey and RK structures which are chains of length $2^\omega$, and they form a hierarchy via projection to barriers of smaller rank.

**Remark.** These results rely on new topological Ramsey spaces and canonization theorems for equivalence relations.
The 2-dimensional Ellentuck space $E_2$

**Goal:** Construct a topological Ramsey space dense in $(\text{Fin} \otimes \text{Fin})^+$. 

**Q.** Which subsets of $(\text{Fin} \otimes \text{Fin})^+$ should we allow? 

**A.** Fix a particular order $\prec$ of the members of non-decreasing sequences of natural numbers of length 2 in order type $\omega$ so that each infinite set is the limit of its finite approximations.

$E_2$ consists of all subsets of $W_2$ for which the $\prec$-preserving bijection is also a tree-isomorphism.

**Figure: $W_2$**
A member of $\mathcal{E}_2$
The collection of $X \subseteq \mathbb{W}_2$ for which the $\prec$-preserving bijection from $\mathbb{W}_2$ to $X$ preserves the tree structure induces the finite approximations. The basic open sets of $\mathcal{E}_2$ are of the form

$$[s, A] = \{ X \in \mathcal{E}_2 : s \sqsubseteq A \sqsubseteq X \}.$$

**Thm.** [D1] $\mathcal{E}_2$ satisfies the 4 axioms of Todorcevic, and hence is a topological Ramsey space: Every subset with the property of Baire is Ramsey.

That $\mathcal{E}_2$ is a topological Ramsey space was heavily utilized when proving the canonization theorem for equivalence relations on barriers on $\mathcal{E}_2$.

This was applied to show that the generic ultrafilter forced by $\mathcal{P}(\omega^2)/\text{Fin} \otimes 2$ has, up to cofinal equivalence, exactly one Tukey type below it, namely that of its projected Ramsey ultrafilter.
The 3-dimensional Ellentuck space $\mathcal{E}_3$

A member of $\mathcal{E}_3$
$\mathcal{E}_S$ for $S$ the Schreier barrier

$X \in \mathcal{E}_S$ only if $X \subseteq \mathbb{W}_S$, for each $n$ for which $X$ has non-empty intersection with the subtree above $(n)$, that restriction of $X$ is in $\mathcal{E}_n$, and more structural requirements which are defined recursively from the structural requirements for $\mathcal{E}_k$. 

Figure: $\mathbb{W}_S$
**Thm.** [D2] For each uniform barrier $B$, there is a topological Ramsey space $E_B$ which is dense in $l_B^+$. Hence, $(E_B, \subseteq^I_B)$ is forcing equivalent to $\mathcal{P}(B)/I_B$.

Thus, the restriction of $\mathcal{P}(B)$ to $E_B$ produces infinitary Ramsey theory, for those partitions into sets satisfying the property of Baire in the Ellentuck topology.

This was a necessary, though not sufficient, step in proving the initial Tukey structures below the ultrafilters $U_B$. 
Other Applications of Extended Ellentuck Spaces
A hierarchy of new Banach spaces

In [Arias, D., Girón, Mijares], we constructed a new Banach spaces using the Tsirelson norm construction over fronts of finite rank on the $\mathcal{E}_k$ spaces.

This forms a hierarchy of spaces over $\ell_p$, with spaces formed from $\mathcal{E}_k$ projecting (in many different ways) to spaces from $\mathcal{E}_j$, for $j < k$.

My motivation for this project was to shed new light on distortion problems. Much work still needs to be done in this direction.
Preservation of ultrafilters by Product Sacks Forcing

**Thm.** [Y.Y. Zheng] The ultrafilters forced by $\mathcal{P}(\omega^k)/\text{Fin} \otimes^k$ are preserved by products of Sacks forcing with countable support.

She first proved a *Moderately-Abstract Parametrized Ellentuck Theorem* for $\mathcal{R} \times \mathbb{R}^\omega$, for a large class of topological Ramsey spaces.

She then showed that $\mathcal{E}_k$ spaces satisfy the premises of this parametrization theorem, which is applied to obtain the theorem above.
A Barren Extension

**Thm.** [Henle, Mathias, Woodin] Let $M$ be a transitive model of $ZF + \omega \rightarrow (\omega)^{\omega}$ and $N$ its Hausdorff extension, that is the extension $M[U]$ where $U$ is the Ramsey ultrafilter forced by $\mathcal{P}(\omega)/\text{Fin}$. Then $M$ and $N$ have the same sets of ordinals; moreover, every sequence in $N$ of elements of $M$ lies in $M$.

In particular, this theorem holds when $M$ is the Solovay model $L(\mathbb{R})$. 
Thm. [D., Hathaway] Fix a uniform barrier $B$. Let $M$ be a transitive model of $\text{ZF} + \text{every subset of } \mathcal{E}_B \text{ is Ramsey}$, and let $N = M[\mathcal{U}_B]$ be the generic extension obtained by forcing with $(\mathcal{E}_B, \subseteq^I_B)$. Then $M$ and $N$ have the same sets of ordinals; moreover, every sequence in $N$ of elements of $M$ lies in $M$.

Thus, there is a hierarchy of models $L(\mathbb{R})[\mathcal{U}_B]$ with stronger and stronger fragments of choice, in the form of containing an ultrafilter $\mathcal{U}_B$ and all $\mathcal{U}_C$ where $C$ is a uniform barrier obtained by a projection of $B$, all of which are barren extensions of $L(\mathbb{R})$. 


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