Banach spaces from barriers in high dimensional Ellentuck spaces

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Spring Topology and Dynamics Conference, 2017
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A Banach space is a complete normed vector space.

c₀ is the space of sequences \((x_n)_{n=1}^{\infty}\) tending toward 0 with the sup norm

\[ \| x \|_\infty = \sup_n |x_n|. \]

\(\ell_p\) is the space of sequences \((x_i)_{i=1}^{\infty}\) such that

\[ \| x \|_p := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty. \]

\(\ell_\infty\) is the space of all bounded sequences with the sup norm.
**Theorem.** (Tsirelson) There is a reflexive Banach space with an unconditional basis not containing $c_0$ or $\ell_p$ for $1 \leq p < \infty$.

Tsirelson’s construction method can be generalized to any barrier.

Argyros and Deliyanni made a systematic study of this construction method for all barriers.

We concentrate on the finite rank barriers $[\omega]^d$ and their compact closures $A_d = [\omega]^{\leq d}$.

For a finite set $E \subseteq \omega$ and a vector $x = \sum_n x_n e_n$,

$$Ex = E \left( \sum_n x_n e_n \right) := \sum_{n \in E} x_n e_n.$$
Low complexity Tsirelson-type spaces $T(\mathcal{A}_d, \theta)$

c₀₀ is the set of all infinite sequences $(x_n)_{n=1}^\infty$ where all but finitely many $x_i$ are zero. Let $1 \leq d < \omega$ and $0 < \theta < 1$. $\mathcal{A}_d = [\omega]^{\leq d}$.

The space $T(\mathcal{A}_d, \theta)$ is the completion of $c₀₀$ with the norm $\| \cdot \|_{T(\mathcal{A}_d, \theta)}$ defined recursively as follows. Given $x = \sum_{n=1}^\infty x_n e_n \in c₀₀$,

$$|x|_0 = \max_n |x_n|.$$ 

$$|x|_{j+1} = \max \left\{ |x|_j, \theta \max \left\{ \sum_{i=1}^m |E_i x|_j : 1 \leq m \leq d \text{ and } E_1 < E_2 < \cdots < E_m \right\} \right\}$$

$$\|x\|_{T(\mathcal{A}_d, \theta)} = \sup_{j < \omega} |x|_j.$$
**Theorem.** (Bellenot) If $d\theta > 1$ and $p$ is such that $d^{1/p} = d\theta$, then for each $x \in T(\mathcal{A}_d, \theta)$,

$$\frac{1}{2d} \|x\|_p \leq \|x\|_{T(\mathcal{A}_d, \theta)} \leq \|x\|_p.$$ 

Thus, $T(\mathcal{A}_d, \theta)$ is isomorphic to $\ell^p$.

$\mathcal{A}_d = [\omega]^{\leq d}$ is the closure under inclusion of the $d$-dimensional barrier on the Ellentuck space.

Motivated by a problem regarding the Tukey structures of ultrafilters, we constructed new topological Ramsey spaces in [D1] and [D2] which turn out to be high dimensional analogues of the Ellentuck space.
$\mathcal{P}(\omega)/\text{Fin}$ forces a \textbf{Ramsey ultrafilter} $\mathcal{U}$.

An equivalent forcing is Mathias forcing mod finite: $([\omega]^{\omega}, \subseteq^*)$.

This is exactly forcing with the Ellentuck space mod finite.

The ideal $\text{Fin} \otimes \text{Fin}$, also denoted $\text{Fin}^{\otimes 2}$, is the set of all $X \subseteq \omega \times \omega$ such that for all but finitely many $n$, the $n$-th fiber of $X$ is finite.

$\mathcal{P}(\omega \times \omega)/\text{Fin}^{\otimes 2}$ forces a non-p-point $\mathcal{U}_2$ satisfying the partition relation

\[
\mathcal{U}_2 \rightarrow (\mathcal{U}_2)_{1,4}^2.
\]

The projection $\pi_1(\mathcal{U}_2)$ to the first coordinates is Ramsey and is generic for $\mathcal{P}(\omega)/\text{Fin}$.
The two dimensional Ellentuck space $\mathcal{E}_2$

In [D1] we constructed a new topological Ramsey space $\mathcal{E}_2$ which is dense in the forcing $\mathcal{P}(\omega \times \omega)/\text{Fin} \otimes^2$.

The purpose was to answer a question left open in [Blass/D./Raghavan] as to the exact Tukey structure below $\mathcal{U}_2$. In [D1] we found that the Tukey structure below $\mathcal{U}_2$ is simply a chain of length 2.

$\mathcal{E}_2$ turned out to be the two dimensional extension of the Ellentuck space. In fact, we constructed $\alpha$-dimensional Ellentuck spaces for each countable ordinal $\alpha$ in [D1] and [D2] which are dense in forcings which produce ever more complex ultrafilters (see [D3]).

As topological Ramsey spaces, they come equipped with notions of barriers (see [Todorcevic]).
The project in this talk was initiated by the following question.

**Question.** (D.) What kinds of Banach spaces can be constructed by extending Tsirelson’s construction to general topological Ramsey spaces?

A natural starting point for answering this question is the extensions of the Ellentuck space to higher dimensions.
Results

\( \mathcal{E}_k \) denotes the \( k \)-dimensional Ellentuck space (defined soon).

For each pair \( 1 \leq d < \omega \) and \( 0 < \theta < 1 \), \( T_k(d, \theta) \) denotes the Banach space constructed from \( \mathcal{E}_k \) using sequences \( E_1 < \cdots < E_d \) of admissible sets (defined soon).

**Theorem.** (ADGM) Given \( k, d, \theta \), \( T_k(d, \theta) \) is a Banach space with the following properties:

1. \( T_k(d, \theta) \) is \( \ell_p \) saturated.
2. \( T_k(d, \theta) \) contains copies of \( \ell^n_\infty \), the bound being the same for all \( n \).
3. For \( j < k \), \( T_j(d, \theta) \) is not isomorphic to \( T_k(d, \theta) \).
4. For \( j < k \), \( T_k(d, \theta) \) contains subspaces isometric to \( T_j(d, \theta) \).
The 2-dimensional Ellentuck space $\mathcal{E}_2$

$\mathcal{E}_2$, $\subseteq \text{Fin}^{\otimes 2}$ is forcing equivalent to $\mathcal{P}(\omega \times \omega)/\text{Fin}^{\otimes 2}$.

$\omega^{\leq 2}$ denotes the set of all non-decreasing sequences of natural numbers of length less than or equal to two.

Let $\prec$ well-order $\omega^{\leq 2}$ in order type $\omega$ as follows:

$$(s_0, s_1) \prec (t_0, t_1) \iff (s_1 < t_1) \text{ or } (s_1 = t_1 \text{ and } s_0 < t_0).$$

Members of $\mathcal{E}_2$ are subsets $X \subseteq \omega^{\leq 2}$ which as trees have the same structure as $\omega^{\leq 2}$ with respect to both tree structure and $\prec$ ordering.
$\omega^{\aleph_0} \leq \omega^2$ has lexicographic order-type $\omega^2$
The well-order $\langle \omega^k \leq 2, \prec \rangle$
The well-order \((\omega^{\lambda \leq 2}, \prec)\)
The well-order \((\omega^\leq 2, \prec)\)
The well-order $\langle \omega^{k \leq 2}, \prec \rangle$
The well-order \((\omega^{\leq 2}, \prec)\)
The well-order \((\omega^{k \leq 2}, \prec)\)
The well-order \((\omega^{2}, \prec)\)
The well-order \((\omega^{\leq 2}, \prec)\)
The well-order \( (\omega^\leq_2, \prec) \)
The well-order \((\omega^2 \leq 2, \prec)\)
The well-order \((\omega_{\leq 2}, \prec)\)
The well-order \((\omega^{\leq 2}, \prec)\)
The well-order \((\omega^{\leq 2}, <)\)
The well-order $(\omega^4, \leq)$
The well-order \( (\omega^{\aleph_0}, \prec) \)
Example: $X \in \mathcal{E}_2$
Example: $X \in \mathcal{E}_2$
Example: $X \in \mathcal{E}_2$
Example: $X \in E_2$
Example: $X \in \mathcal{E}_2$
Example: $X \in \mathcal{E}_2$
Example: $X \in \mathcal{E}_2$
Example: $X \in \mathcal{E}_2$
Example: $X \in \mathcal{E}_2$
The pink dots are a 6-th approximation to $X \in \mathcal{E}_2$. 
A 6-th approximation to a different member of $\mathcal{E}_2$.

The collection of all these types of sets is the notion for $\mathcal{E}_2$ corresponding to the barrier $[\omega]^6$ in the Ellentuck space.
Finite dimensional barriers on $\mathcal{E}_2$

The $i$-th dimensional barrier $\mathcal{AR}_i^2$ on $\mathcal{E}_2$ is the collection of all $i$-th approximations of members of $\mathcal{E}_2$.

This is the analogue for $\mathcal{E}_2$ of the $i$-dimensional barrier $[\omega]^i$ on the Ellentuck space.

$$\mathcal{AR}^2 = \bigcup_{i<\omega} \mathcal{AR}_i^2.$$ 

There are several choices to make for extending Tsirelson’s construction to high dimensional Ellentuck spaces:

The admissible sets could be in the closure of a barrier or simply finite sets.

The endpoints separating the sequence of admissible sets could be either in $\bigcup_{m\leq d} \mathcal{AR}_m^2$ or simply finite sets of size $\leq d$. 
The Banach space $T_2(d, \theta)$

We define $T_2(d, \theta)$ as the completion of $c_{00}$ under the norm defined using sequences

$$v_1 \preceq E_1 \prec \cdots \prec v_m \preceq E_m$$

where $m \leq d$, the collection $\{v_i : 1 \leq i \leq m\}$ is simply a set, and each $E_i \in AR^2$. 
The basis for the space consists of the non-decreasing sequences of length two, ordered by $\prec$. Denote these as $e_n$, $n < \omega$.

Let $x = \sum_{n=1}^{\tilde{n}} x_n e_n$ be a member of $c_{00}$. Define $|x|_0 = \max_n |x_n|$.

$$|x|_{j+1} = \max \left\{ |x|_j, \theta \max \left\{ \sum_{i=1}^m |E_i x|_j : (E_i)_{i=1}^m \text{ is admissible}, \ m \leq d \right\} \right\}.$$  

Define $\|x\|_{T(A_d, \theta)} = \sup_{j < \omega} |x|_j$. 


Theorem. (ADGM) Given $k, d, \theta$, $T_k(d, \theta)$ is a Banach space with the following properties:

1. $T_k(d, \theta)$ is $\ell_p$ saturated.
2. $T_k(d, \theta)$ contains copies of $\ell_n^\infty$, the bound being the same for all $n$.
3. For $j < k$, $T_j(d, \theta)$ is not isomorphic to $T_k(d, \theta)$.
4. For $j < k$, $T_k(d, \theta)$ contains subspaces isometric to $T_j(d, \theta)$. 
$T_1(d, \theta)$ embeds isometrically as a subspace of $T_2(d, \theta)$

Let $T_2(\theta, d)[(0)]$ be the subspace of $T_2(\theta, d)$ generated by the basis elements $\{(0, n) : n < \omega\}$.

Given $y = \sum_{n=1}^{\omega} x_n(0, j_n)$, let $\varphi(y) = \text{tr}_0(y) = \sum_{n=1}^{\omega} x_n(j_n)$. Then

$$\|y\|_{T_2(\theta, d)[(0)]} = \|\varphi(y)\|_{T_1(\theta, d)}.$$ 

This follows from the fact that the trace above $(0)$ of any admissible sequence in $E_2$ is an admissible sequence in $E_1$.

Thus, $\varphi$ maps $T_2(\theta, d)[(0)]$ isometrically to $T_1(\theta, d)$.

This is the idea behind the more general result that $T_{k+1}(d, \theta)$ contains a subspace isometric to $T_k(d, \theta)$. 
Ex. The norms on $T_1(2, 3/4)$ and $T_2(2, 3/4)$ are different

Let $x = 2(0, 0) + 1(1, 1) + 1(0, 2)$ in $T_2(2, 3/4)$. Then

$$\|x\|_{T_2(2,3/4)} = |x|_2 = \frac{3}{4} \left( \frac{3}{4} (2 + 1) + 1 \right) = \frac{39}{16}.$$

Let $z = 2e_i + e_k + e_l$ in $T_1(2, 3/4)$ for any $i < k < l$. Then

$$\|x\|_{T_1(2,3/4)} = |x|_2 = \frac{3}{4} \left( 2 + \frac{3}{4} (1 + 1) \right) = \frac{21}{8}.$$

Thus, no vector in $T_1(2, 3/4)$ formed with three basis elements and the coefficients 2, 1, 1 has norm equal to $\|x\|_{T_2(2,3/4)}$.

In general, the norm for any vector in $T_{k+1}(d, \theta)$ is no larger than the norm for a similar vector in $T_k(d, \theta)$. 
The 3-dimensional Ellentuck space $\mathcal{E}_3$

The 3-dimensional Ellentuck space is forcing equivalent to $\mathcal{P}(\omega^3)/\text{Fin}^\otimes 3$.

This produces an ultrafilter $\mathcal{U}_3$ which projects to the ultrafilters $\mathcal{U}_2$ and the Ramsey ultrafilter $\mathcal{U}_3$.

The space $\mathcal{E}_3$ was built in order to obtain a precise analysis of the Rudin-Keisler and Tukey structures below $\mathcal{U}_3$. 
The structure of members in $\mathcal{E}_3$
The structure of members in $\mathcal{E}_3$
The structure of members in $E_3$
The structure of members in $\mathcal{E}_3$

Diagram:

0 → 1 → 2 → 0
1 → 2 → 3 → 1
2 → 3 → 4 → 2
1 → 2 → 3 → 1
2 → 3 → 4 → 2
3 → 4 → 5 → 3
4 → 5 → 6 → 4
The structure of members in $\mathcal{E}_3$
The structure of members in $\mathcal{E}_3$
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The structure of members in $E_3$
The structure of members in $\mathcal{E}_3$

The $\mathcal{E}_k$ for $k \geq 3$ continues this kind of structure. These topological Ramsey spaces are forcing equivalent to $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$, adding a hierarchy of non-p-point ultrafilters satisfying weaker partition relations.
Future Directions

Given that the norms the $T_k(d, \theta)$ are non-increasing and, in some places, decreasing as $k$ increases, it seems plausible that more general spaces, using higher dimensional Ellentuck spaces and/or higher dimensional barriers, could provide interesting examples regarding distortion.

What other properties or new types of Banach spaces and be obtained by applying the barrier construction of a norm?

How do the Banach spaces differ depending on which of the three possible combinations for endpoints and admissible sets are chosen for defining the norm?
References


