A Survey of Osborn Loops

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Osborn's Paper

Osborn studied isotopy invariance of the *weak inverse property* (WIP):

$$x(yx)^\rho = y^\rho \quad \text{or} \quad (xy)^\lambda x = y^\lambda$$

He showed that if WIP is universal then this identity holds:

$$(*) \qquad x(yz \cdot x) = (x \cdot yE_x) \cdot zx$$

where E_x is a permutation.

Thm: If Q is a WIP loop satisfying (*), then Q/N is a Moufang loop.

Basarab Steps In

Basarab dubbed a loop satisfying any of the following equivalent identities an *Osborn loop*:

$$
\begin{aligned}
x(yz \cdot x) &= (x \cdot yE_x) \cdot zx \\
x(yz \cdot x) &= (x^\lambda \backslash y) \cdot zx \\
(x \cdot yz)x &= xy \cdot (zE_x^{-1} \cdot x) \\
(x \cdot yz)x &= xy \cdot (z/x^\rho)
\end{aligned}
$$

Here

$$
\begin{aligned}
E_x &= R_x R_{x^\rho} = (L_x L_{x^\lambda})^{-1} \\
&= R_x L_x R_x^{-1} L_x^{-1}
\end{aligned}
$$

Obviously every Moufang loop is an Osborn loop.

Autotopic Characterizations:

For each x, there is a permutation A_x such that

$$(A_x, R_x, R_x L_x)$$

is an autotopism. (It follows that $A_x = E_x L_x$.)

OR

For each x, there is a permutation B_x such that

$$(L_x, B_x, L_x R_x)$$

is an autotopism. (It follows that $B_x = E_x^{-1} R_x$.)

Every CC-loop is Osborn

(Probably due to Basarab)

Prf: Multiply the CC autotopisms

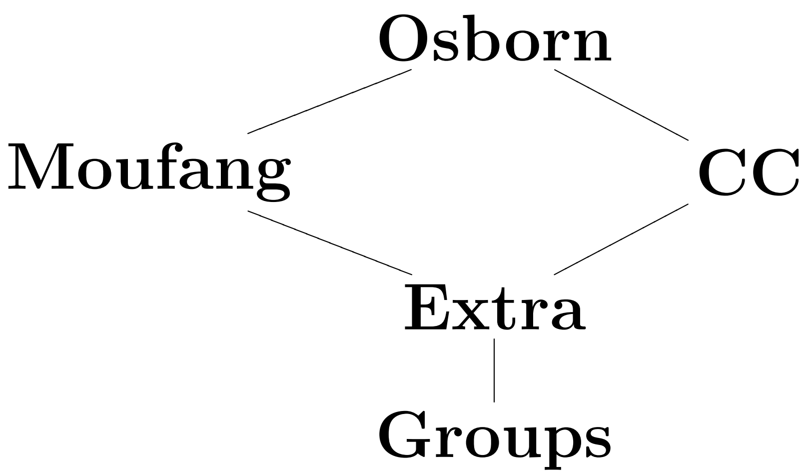
$$\begin{aligned} RCC &: (R_x, R_x L_x^{-1}, R_x) \\ LCC &: (L_x R_x^{-1}, L_x, L_x) \end{aligned}$$

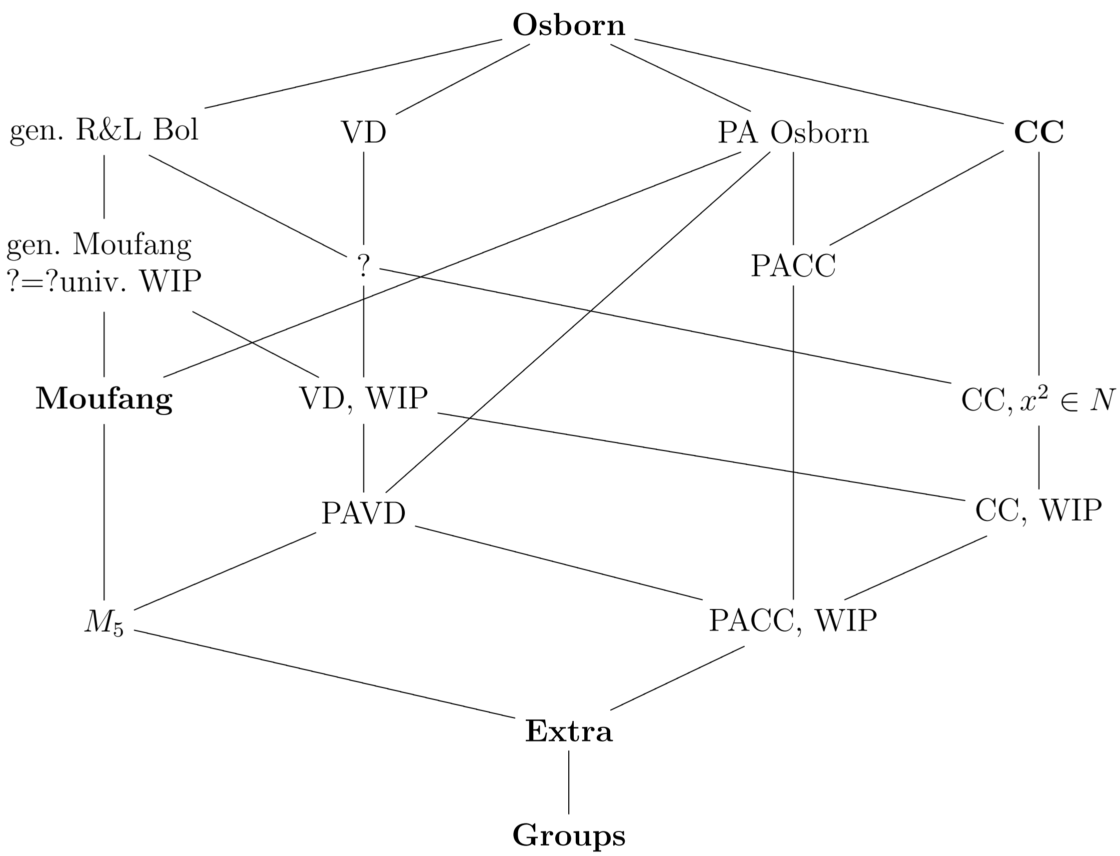
to get

$$(R_x L_x R_x^{-1}, R_x, R_x L_x).$$

More generally:

Thm: Let Q be a G-loop, i.e., for each $x \in Q$, there is
a left pseudoautomorphism F_x and
a right pseudoautomorphism G_x,
each with companion x. Suppose also that $F_x G_x = G_x F_x = I$ and $xF_x = xG_x = x$. Then Q is an Osborn loop.


Osborn
Moufang
CC
Extra
Groups



VD: (Basarab)
(i) Each T_x is a *right* pseudoaut. with companion x.
(ii) Each T_x^{-1} is a *left* pseudoaut. with comp. x.

Generalized Moufang: (Basarab) [equivalent to "Osborn and WIP"]

$$x(yz\cdot x) = (y^\lambda x^\lambda)^\rho \cdot zx \quad \text{or} \quad (x\cdot zy)x = xz\cdot(x^\rho y^\rho)^\lambda$$

Generalized Right & Left Bol: (Belousov)

$$zy\cdot x = zx^\rho\cdot(xy\cdot x) \qquad \text{and} \qquad x\cdot yz = (x\cdot yx)\cdot x^\lambda z$$

$\mathbf{M_5}$: (Pflugfelder) Moufang and each x^4 is nuclear.

Elementary Properties:

It does not take much to force an Osborn loop to be Moufang: LAlt, RAlt, Flex, LIP, RIP, AAIP, etc.

In particular, a commutative Osborn loop is a commutative Moufang loop.

By contrast, you cannot force an Osborn loop to be CC with a two variable identity. In other words, there is no identity φ in two variables such that:

- φ holds in all CC-loops, and
- an Osborn loop satisfying φ is CC.

This is because every identity in two variables holding in all CC-loops also holds in all Moufang loops.

More Elementary Properties:

Let Q be an Osborn loop.

- (Basarab) The three nuclei coincide, and the nucleus is normal.

- Every right inner mapping is a right pseudoautomorphism. The companion of $R(x, y) = R_x R_y R_{xy}^{-1}$ is $(xy)^\lambda(y^\lambda \backslash x)$. Similarly, left inner mappings are left pseudoautomorphisms.

- **Coro:** If Q is an $A_{r,l}$ Osborn loop, then Q/N is a commutative Moufang loop.

This gives us yet another proof of

Basarab's CC-loop Thm: The quotient of a CC-loop by its nucleus is an abelian group.

Still More Elementary Stuff:

• The left and right inner mapping groups coincide. Indeed,

$$R(x,y)^{-1} = [L_{y^\rho}^{-1}, R_x^{-1}] = L(y^\lambda, x^\lambda)$$

This is a triviality for Moufang loops. For CC-loops, it was first observed by Drápal.

Still mysterious are the *middle* inner mappings $T_x = R_x L_x^{-1}$.

In a Moufang loop, each T_x is a *right* pseudoaut. with companion x^{-3}.

In a CC-loop, each T_x is a *left* pseudoaut. with companion x.

Can one say something in arbitrary Osborn loops?

Basarab's Osborn Loop Thm:

Let Q be a *universal* Osborn loop, i.e., every isotope is Osborn.

Then Q/N has WIP.

And so $(Q/N)/N(Q/N)$ is Moufang (by Osborn).

Thus if N_2 is the *second nucleus* of Q, then Q/N_2 is a Moufang loop.

Coro: A simple universal Osborn loop is Moufang.

Open Problem:

Is every Osborn loop universal?

If not, does there exist a proper Osborn loop with trivial nucleus?

If not every Osborn loop is universal, does there exist a "nice" identity characterizing universal Osborn loops?

Trivial Nuclei:

$$\mathcal{A} := \left\langle \left(L_{x^\lambda}^{-1}, R_x, R_x L_x \right) : x \in Q \right\rangle.$$

Group epimorphisms:

$$\begin{aligned}
\mathcal{A} &\rightarrow \mathrm{RMlt}(Q) : (\alpha, \beta, \gamma) \mapsto \beta \\
\mathcal{A} &\rightarrow \mathrm{LMlt}(Q) : (\alpha, \beta, \gamma) \mapsto \alpha \\
\mathcal{A} &\rightarrow \mathrm{PMlt}(Q) : (\alpha, \beta, \gamma) \mapsto \gamma
\end{aligned}$$

The kernels are isomorphic to subgroups of the nucleus. So if the nucleus is trivial, then

$$\mathrm{RMlt}(Q) \cong \mathrm{LMlt}(Q) \cong \mathrm{PMlt}(Q).$$

On generators,

$$R_x \leftrightarrow L_{x^\lambda}^{-1} \leftrightarrow R_x L_x.$$

If the isomorphisms extended to automorphisms of $\mathrm{Mlt}(Q)$, we would have some ingredients of triality.

Examples:

The smallest order for which proper (nonMoufang, nonCC) Osborn loops with nontrivial nucleus exist is 16. There are two such loops.

• Each of the two is a G-loop, i.e., isomorphic to all isotopes.

• Each contains as a subgroup the dihedral group of order 8.

• For each loop, the center coincides with the nucleus and has order 2. The quotient by the center is a nonassociative CC-loop of order 8.

• The second center is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the quotient is $\mathbb{Z}_4$.

• One loop satisfies $L_x^4 = R_x^4 = I$, the other does not.

Miscellanea:

• In an Osborn loop Q, the semicenter/commutant/centrum/etc.

$$C(Q) = \{a \in Q : ax = xa,\ \forall x \in Q\}$$

is a subloop. It is not necessarily normal, of course. (Recall that an open problem in Moufang loops is to exhibit explicitly a Moufang loop in which $C(Q)$ is not normal.)

• If T_a is an automorphism, then $a \cdot aa = aa \cdot a \in N$. Thus for every $a \in C(Q)$, we have $a^3 \in Z(Q)$.

• If $(xx)^\rho = x^\rho x^\rho$ holds, then

$$x^{\rho\rho\rho\rho\rho\rho} = x.$$

Osborn Loops with AIP:

Automorphic Inverse Property (AIP):

$$(xy)^\rho = x^\rho y^\rho \quad \text{or} \quad (xy)^\lambda = x^\lambda y^\lambda$$

AIP Osborn loops include:

- commutative Moufang loops
- AIP CC-loops

The smallest AIP CC-loops have order 9. So the smallest *known* proper AIP Osborn loops have order 729.

The smallest AIP PACC-loops have order 27. So the smallest *known* proper AIP PA Osborn loops have order 2187.

Surely there are smaller examples.

AIP Osborn Loops II:

Thm: In an AIP Osborn loop, the cubing maps

$$x \mapsto x \cdot xx \qquad \text{and} \qquad x \mapsto xx \cdot x$$

are centralizing endomorphisms.

This generalizes the classic foundational result of CML theory:

Coro: In a CML, the cubing map $x \mapsto x^3$ is a centralizing endomorphism.

Coro: If Q is an AIP CC-loop, then Q/Z is an elementary abelian 3-group.

AIP Osborn Loops III:

Coro: If Q is an AIP, universal Osborn loop, then Q/Z is a commutative Moufang loop of exponent 3.

Coro: If Q is a finitely generated, AIP, universal Osborn loop, then Q is centrally nilpotent. If a minimal generating set for Q has $n > 1$ elements, then the nilpotence class of Q is at most n.

Commutative Moufang Loops of Exponent 3 — **Symmetric, Distributive Quasigroups**

Hall Triple Systems

$$(Q, \circ) \text{ loop, } (Q, \cdot) \text{ quasigroup}$$

$$x \cdot y = (x \circ x) \circ (y \circ y)$$

$$x \circ y = (x/e) \cdot (e \backslash y)$$

AIP Osborn Loops of Exponent 3 — **New Variety of Quasigroups**

???

New Quasigroups

The new variety of quasigroups is defined by the equations

$$(x \cdot yz)(zx \cdot y) = z$$

and

$$x(x \cdot xy) = y(y \cdot yx)$$

(or its mirror).

Properties:

$$\text{Idempotence: } xx = x$$

$$L_x^2 R_x^2 = L_x^6 = R_x^6 = I$$

$$(L_x L_y)^3 = (R_x R_y)^3 = I$$

$$L_x^2, R_x^2 \in \mathrm{Aut}(Q)$$

So the multiplication group is some sort of generalized Fisher group.

Conjugacy Closed Quasigroups

A quasigroup is *conjugacy closed* if the sets of left and right translations are each closed under self-conjugation: for each x, y, there exist u, v such that

$$L_x^{-1} L_y L_x = L_u \quad \text{and}$$
$$R_x^{-1} R_y R_x = R_v$$

CC-quasigroups include:

- CC-loops
- Quasigroups isotopic to groups
- Trimedial quasigroups

Conjecture: Every CC-quasigroup is isotopic to an Osborn loop.