Fine gradings and gradings by root systems on simple Lie algebras

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Gradings

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\( G \) abelian group, \( \mathcal{A} \) algebra over \( \mathbb{F} \).

(Only finite-dimensional algebras will be considered here.)
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**\( G \)-grading on \( \mathcal{A} \):**

\[
\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,
\]

\[
\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{g+h} \quad \forall g, h \in G.
\]
Universal group

Universal group: This is the group \((U(\Gamma), \ast)\) generated by \(\text{Supp}\, \Gamma\) subject to the relations \(g \ast h = g + h\) for any \(g, h \in \text{Supp}\, \Gamma\) such that \(g + h \in \text{Supp}\, \Gamma\):

\[
U(\Gamma) := \langle \text{Supp}\, \Gamma \rangle / \langle g \ast h \ast (-(g + h)) \rangle : g, h, g + h \in \text{Supp}\, \Gamma.
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\(\Gamma\) can then be realized as a grading by \(U(\Gamma)\).
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Example

\[ \mathcal{L} = \mathfrak{sl}_2(\mathbb{F}) = \text{span}\{ e = (0,1,0), \ h = (1,0,0), \ f = (0,0,1) \}. \]
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\[ \Gamma : \mathcal{L} = \mathbb{F}e \oplus \mathbb{F}h \oplus \mathbb{F}f \quad \mathbb{Z}_3\text{-grading} \]

\[ \uparrow \quad \uparrow \quad \uparrow \]

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\[ U(\Gamma) = \mathbb{Z}, \text{ because } [e, e] = [f, f] = 0. \]
Fine gradings

\[ \Gamma : A = \bigoplus_{g \in G} A_g, \quad \Gamma' : A = \bigoplus_{g' \in G'} A'_{g'}, \quad \text{gradings on } A. \]
Fine gradings

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}'_{g'}, \quad \text{gradings on } \mathcal{A}. \]

- \( \Gamma \) is a refinement of \( \Gamma' \) if for any \( g \in G \) there is a \( g' \in G' \) such that \( \mathcal{A}_g \subseteq \mathcal{A}_{g'} \).

Remark: Any grading is a coarsening of a fine grading.
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- \( \Gamma \) is fine if it admits no proper refinement.
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**Remark**

Any grading is a coarsening of a fine grading.
Example: Cartan grading

$$g = h \oplus (\bigoplus_{\alpha \in \Phi} g_{\alpha})$$

(root space decomposition of a semisimple complex Lie algebra).

This is a fine grading by $$\mathbb{Z} \Phi \cong \mathbb{Z}^n$$, $$n = \text{rank } g$$. 

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Example: Pauli matrices
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\[ A = \text{Mat}_n(\mathbb{F}) \]

\[
X = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & \epsilon & 0 & \ldots & 0 \\
0 & 0 & \epsilon^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \epsilon^{n-1}
\end{pmatrix}
\]

\[
Y = \begin{pmatrix}
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\end{pmatrix}
\]

(\(\epsilon\) a primitive \(n\)th root of 1)

\[ X^n = 1 = Y^n, \quad YX = \epsilon XY \]

\[ \mathcal{A} = \bigoplus_{(\bar{i}, \bar{j}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}(\bar{i}, \bar{j}), \quad \mathcal{A}(\bar{i}, \bar{j}) = \mathbb{F} X^i Y^j. \]
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\(A\) becomes a graded division algebra.

This grading induces a fine grading on \(\mathfrak{sl}_n(\mathbb{F})\):
\[ \mathfrak{sl}_n(\mathbb{F}) = \bigoplus_{0 \neq (\bar{i}, \bar{j}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathbb{F} X^i Y^j. \]
Example: Octonions

The Cayley algebra $O$ is obtained from the ground field $F$ by means of the Cayley-Dickson doubling process:

- $K = F \oplus F$, $Z_2$-graded;
- $H = K \oplus K$, $Z_2^2$-graded;
- $O = H \oplus H$, $Z_3^2$-graded.

All these are fine gradings.
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- \( \mathcal{O} = H \oplus H \mathbf{l} \), \( \mathbb{Z}_2^3 \)-graded.

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Gradings

Gradings by root systems

Fine gradings and gradings by root systems
Gradings by root systems

Definition (Berman-Moody)

A Lie algebra $L$ over $\mathbb{F}$ is graded by the reduced root system $\Phi$, or $\Phi$-graded, if:

1. $L$ contains as a subalgebra a finite-dimensional semisimple Lie algebra $g = h \oplus (\bigoplus_{\alpha \in \Phi} g_{\alpha})$ whose root system is $\Phi$ relative to a Cartan subalgebra $h = g_0$;
2. $L = \bigoplus_{\alpha \in \Phi \cup \{0\}} L(\alpha)$, where $L(\alpha) = \{x \in L : [h, x] = \alpha(h)x \}$ for all $H \in h$;
3. $L(0) = \sum_{\alpha \in \Phi} [L(\alpha), L(-\alpha)]$. The subalgebra $g$ is said to be a grading subalgebra of $L$. 

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The subalgebra \( g \) is said to be a \textit{grading subalgebra} of \( \mathcal{L} \).
Gradings by root systems

For irreducible $\Phi$, view $\mathcal{L}$ as a module for $g$. As such it is a direct sum of copies of the adjoint, the little adjoint and the trivial modules. We may collect isomorphic irreducible $g$-submodules in $\mathcal{L}$:

$$\mathcal{L} = (g \otimes A) \oplus (W \otimes B) \oplus D,$$

where $A$ is a grading subalgebra identified with $g \otimes 1$ for a distinguished element $1 \in A$, $W$ is $0$ if $\Phi$ is simply laced, while $W$ is the little adjoint module (the irreducible $g$-module whose highest weight is the highest short root) otherwise, $D$ is the centralizer of $g \cong g \otimes 1$, and hence it is a subalgebra of $\mathcal{L}$. 
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- the grading subalgebra $\mathfrak{g}$ is identified with $\mathfrak{g} \otimes 1$ for a distinguished element $1 \in \mathcal{A}$,
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- $\mathcal{D}$ is the centralizer of $\mathfrak{g} \simeq \mathfrak{g} \otimes 1$, and hence it is a subalgebra of $\mathcal{L}$. 


Coordinate algebra

The Lie bracket in $\mathfrak{L}$ induces a multiplication on the sum $a = A \oplus B$. $a$ becomes a unital nonassociative algebra: the coordinate algebra. Associative, alternative, Jordan and structurable algebras appear as coordinate algebras. The elements of the Lie subalgebra $\mathfrak{D}$ act as derivations on $a$. 
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Let $\mathcal{O}$ be the Cayley algebra over $\mathbb{F}$, and let $\mathcal{J}$ be a central simple degree 3 Jordan algebra.

Remark
An extension of Tits construction gives, up to isomorphisms, all $G_2$-graded Lie algebras (Benkart-Zelmanov).
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Let $\mathcal{O}$ be the Cayley algebra over $\mathbb{F}$, and let $\mathcal{J}$ be a central simple degree 3 Jordan algebra.

Consider Tits' construction:

$$\mathcal{T}(\mathcal{O}, \mathcal{J}) = \text{der} \mathcal{O} \oplus (\mathcal{O}_0 \otimes \mathcal{J}_0) \oplus \text{der} \mathcal{J}.$$ 

Here $\mathfrak{g} = \text{der} \mathcal{O}$ is the simple Lie algebra of type $G_2$, $\mathcal{W} = \mathcal{O}_0$ is its little adjoint module.
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$\mathcal{T}(\mathcal{O}, \mathcal{J})$ is graded by the root system $G_2$, with coordinate algebra $\mathcal{J}$. 

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Nonreduced root systems

Berman-Moody's definition can be extended to cover nonreduced root systems, thus considering, in the irreducible case, $BC_r$-graded Lie algebras (Benkart-Smirnov, Allison-Benkart-Gao). An extra summand appears in the decomposition into isotypical components:

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$$\mathcal{L} = (g \otimes A) \oplus (\mathcal{W} \otimes B) \oplus (\mathcal{V} \otimes C) \oplus D,$$

where $\mathcal{L}$ is the Lie algebra and $g$, $A$, $\mathcal{W}$, $\mathcal{V}$, $\mathcal{C}$, $B$, and $\mathcal{D}$ are other algebraic objects.
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The coordinate algebra is then $\mathfrak{a} = A \oplus B \oplus C$. 
Gradings

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Some properties of fine gradings

Proposition

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group $G$. Then:

▶ The neutral homogeneous component $\mathcal{L}_0$ is a toral subalgebra of $\mathcal{L}$ (i.e., $\text{ad} \mathcal{L}_0$ consists of commuting diagonalizable operators in $\mathcal{L}$).

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Proposition (continued)

\[ \text{Let } \text{tor}(G) \text{ be the torsion subgroup of } G. \text{ The coarsening } \bar{\Gamma} : \bar{L} = \bigoplus_{\bar{g} \in G/\text{tor}(G)} \bar{L}_{\bar{g}}, \text{ is the weight space decomposition of } L \text{ relative to } L_0. \text{ That is, for any } \bar{g} \in \text{Supp} \bar{\Gamma}, \text{ there is a linear form } \alpha \in L^* \text{ such that } L_{\bar{g}} = L(\alpha) = \{ x \in L : [h, x] = \alpha(h) x \forall h \in L_0 \}. \]
Let $\text{tor}(G)$ be the torsion subgroup of $G$. The coarsening

$\Gamma : \mathcal{L} = \bigoplus_{\bar{g} \in G/\text{tor}(G)} \bar{\mathcal{L}}_{\bar{g}}$, 

where $\bar{\mathcal{L}}_{\bar{g}} = \bigoplus_{h \in \text{tor}(G)} \mathcal{L}_{g+h}$, is the weight space decomposition of $\mathcal{L}$ relative to $\mathcal{L}_0$. 


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where \( \bar{L}_{\bar{g}} = \bigoplus_{h \in \text{tor}(G)} L_{g+h} \), is the weight space decomposition of \( L \) relative to \( L_0 \).

That is, for any \( \bar{g} \in \text{Supp} \Gamma \), there is a linear form \( \alpha \in L_0^* \) such that \( L_{\bar{g}} \) equals

\[
L(\alpha) = \{ x \in L : [h, x] = \alpha(h)x \ \forall h \in L_0 \}.
\]
Moreover,

The set \( \Phi = \{ \alpha \in L^* \cup \{0\}: L_\alpha \neq 0 \} \) is a (possibly nonreduced) irreducible root system.

The map \( \pi: G \to \mathbb{Z} \Phi \) such that \( L_g \subseteq L_\alpha \), is a surjective group homomorphism, with \( \ker \pi = \text{tor}(G) \).
Moreover,

**Proposition**

- The set
  \[ \Phi = \{ \alpha \in \mathcal{L}_0^* \setminus \{0\} : \mathcal{L}(\alpha) \neq 0 \} \]

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**Proposition**

- **The set**

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is a *(possibly nonreduced)* irreducible root system.

- **The map**

\[ \pi : G \rightarrow \mathbb{Z}\Phi \]

\[ g \mapsto \alpha \quad \text{such that } \mathcal{L}_g \subseteq \mathcal{L}(\alpha), \]

is a surjective group homomorphism, with \( \ker \pi = \text{tor}(G) \).
Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group $G$.

Let $\Phi$ be the associated root system.

Let $\tilde{G}$ be a complement of $\text{tor}(G)$: $G = \tilde{G} \oplus \text{tor}(G)$, and consider the subalgebra $g = \bigoplus_{g \in \tilde{G}} \mathcal{L}_g$. 
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**Theorem**

$\mathcal{L}$ is graded by the root system $\Phi$ with grading subalgebra $g$. 

Grading on the coordinate algebra

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group $G$. 
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Grading on the coordinate algebra

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Then \( \Gamma \) induces:

(i) a grading by the irreducible root system \( \Phi \),

(ii) a fine grading by \( \text{tor}(G) \) on the coordinate algebra \( \mathfrak{a} \), which satisfies \( \mathfrak{a}_0 = \mathbb{F}1 \).
Examples

The fine gradings on the exceptional simple Lie algebras such that the free rank of its universal group is $\geq 3$ are the following:
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The fine gradings on the exceptional simple Lie algebras such that the free rank of its universal group is \( \geq 3 \) are the following:

- The Cartan gradings on \( F_4, E_6, E_7 \) and \( E_8 \).
Examples

The fine gradings on the exceptional simple Lie algebras such that the free rank of its universal group is $\geq 3$ are the following:

- The Cartan gradings on $F_4$, $E_6$, $E_7$ and $E_8$.
- A fine grading on $E_7$ by $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ related to a grading by the root system $C_3$:

$$e_7 = T(\mathcal{O}, \mathcal{H}_3(\mathbb{H})) = \text{der} \mathcal{O} \oplus (\mathcal{O}_0 \otimes \mathcal{H}_3(\mathbb{H})_0) \oplus \text{der} \mathcal{H}_3(\mathbb{H}).$$

Here $\text{der} \mathcal{H}_3(\mathbb{H})$ is the simple Lie algebra of type $C_3$, and $\mathcal{H}_3(\mathbb{H})_0$ is its little adjoint module. The coordinate algebra is $\mathcal{O}$, endowed with its $\mathbb{Z}_2^3$-grading.
Examples

Gradings by $\mathbb{Z}^4 \times \mathbb{Z}^{r-5}$ on $E_r$ ($r = 6, 7, 8$) related to gradings by the root system $F_4$:

$$e_r = \mathcal{T}(C, \mathcal{H}_3(\mathbb{O})) = \text{der} C \oplus (C_0 \otimes \mathcal{H}_3(\mathbb{O})_0) \oplus \text{der} \mathcal{H}_3(\mathbb{O}).$$

Here $\text{der} \mathcal{H}_3(\mathbb{O})$ is the simple Lie algebra of type $F_4$, and $\mathcal{H}_3(\mathbb{O})_0$ is its little adjoint module.

The coordinate algebra is $C = \mathbb{K}, \mathbb{H}$ or $\mathbb{O}$ endowed, respectively, with its fine grading by $\mathbb{Z}_2, \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$. 

Classification of fine gradings

The fine gradings on simple Lie algebras with infinite universal groups are thus obtained by combining a grading by a root system and a 'special grading' on the coordinate algebra.

The classification of the fine gradings on the classical simple Lie algebras was completed in 2010, on $G_2$ in 2006 (Draper–Martín-González, and independently Bahturin–Tvalavadze), on $F_4$ in 2009 (Draper–Martín-González), and on $E_6$ in 2012 (Draper–Viruel, preprint).

A whole bunch of fine gradings has been obtained, using the relationship of fine gradings and gradings by root systems, for the exceptional simple Lie algebras $E_7$ and $E_8$, but the classification of the fine gradings for these Lie algebras is not yet complete.
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That’s all. Thanks