Nuclear semidirect product of commutative automorphic loops

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Semidirect product of groups

Fact (Semidirect product as a configuration)

Let $G$ be a group and let $H < G$ and $K \triangleleft G$ such that $KH = G$ and $K \cap H = 1$. Then $G$ is a semidirect product of $K$ and $H$, denoted by $G = K \rtimes H$.

Fact (Semidirect product as a construction)

Let $K, H$ be two groups and $\varphi : H \to \text{Aut}(K)$ a homomorphism. Then the set $K \times H$ equipped with the binary operation

$$(a, i) \ast (b, j) = (a \cdot \varphi_i(b), i \cdot j)$$

is a group, denoted by $K \rtimes_\varphi H$.

Fact (The correspondence)

$K \times 1$ is a normal subgroup and $1 \times H$ is a subgroup of $K \rtimes_\varphi H$. On the other hand, starting with $G$, we can define $\varphi_i$ as $k \mapsto k^i$. 
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**Definition**

A loop $Q$ is called *automorphic* if $\text{Inn}(Q) \subseteq \text{Aut}(Q)$.

**Fact**

Let $Q$ be a commutative loop. Then $\text{Inn}(Q) = \langle L_{x,y}; \, x, y \in Q \rangle$, where $L_{x,y} = L_{xy}^{-1}L_xL_y$.

**Corollary**

A commutative loop $Q$ is automorphic if and only if, for all $x, y, u, v \in Q$,

$\frac{(uv \cdot x) \cdot y}{xy} = \frac{(ux \cdot y)}{xy} \cdot \frac{(vx \cdot y)}{xy}$. 


Commutative automorphic loops

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Nuclear semidirect product

Let \((Q, +)\) be a commutative automorphic loop. We consider subloops \(H\) and \(K\) of \(Q\) such that

- \(K + H = Q\) and \(K \cap H = \{0\}\);
- \(K \triangleleft H\);
- \(K\) and \(H\) are abelian groups;
- \(K \leq N_\mu(Q)\).

Example

Let \(Q\) be the non-associative commutative Moufang loop with 81 elements. \(Q\) is of exponent 3 and there exists a normal subgroup of order 27 and hence \(Q \cong \mathbb{Z}_3^3 \rtimes \mathbb{Z}_3\). However \(N(Q) \cong \mathbb{Z}_3\).

Lemma

If \(a, b \in K\) and \(i, j \in H\) as above then
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(a + i) + (b + j) = L_{ij}(a + b) + (i + j).
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**Lemma**

If \(a, b \in K\) and \(i, j \in H\) as above then

\[(a + i) + (b + j) = L_{i,j}(a + b) + (i + j)\].
External semidirect product

Proposition

Let $H$ and $K$ be two abelian groups and let $\varphi$ be a mapping $\varphi : H^2 \to \text{Aut}(K)$. We define an operation $\ast$ on $Q = K \times H$ as follows:

$$(a, i) \ast (b, j) = (\varphi_{i,j}(a + b), i + j).$$

Then $Q$ is a commutative automorphic loop if and only if

1. $\varphi_{i,j} = \varphi_{j,i}$;
2. $\varphi_{i,0} = \text{id}_K$;
3. $\varphi_{i,j} \circ \varphi_{k,n} = \varphi_{k,n} \circ \varphi_{i,j}$;
4. $\varphi_{i,j,k} = \varphi_{j,k,i} = \varphi_{k,i,j}$;
5. $\varphi_{i,j+k} + \varphi_{j,i+k} + \varphi_{k,i+j} = \text{id}_K + 2\varphi_{i,j,k}$;

for all $i, j, k, n \in H$, where $\varphi_{i,j,k} = \varphi_{i,j+k} \circ \varphi_{j,k}$. 
[Q : K] = 2

**Example**

Let $H \cong \mathbb{Z}_2$. Then

$$\varphi_{0,0} = \varphi_{1,0} = \varphi_{0,1} = \text{id}_K.$$  

The only other non-trivial condition is

$$\varphi_{1,0} + \varphi_{1,0} + \varphi_{1,0} = \text{id}_K + 2\varphi_{1,1,1} \quad 3 \text{id}_K = \text{id}_K + 2 \text{id}_K \circ \varphi_{1,1} \quad 2 \text{id}_K = 2\varphi_{1,1}$$

In other words, $\varphi_{1,1}(2x) = 2x$. 
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Proposition

Let $M$ be a faithful module over a ring $R$, $2 \in R^*$, and let $r \in R^*$ be of a multiplicative order $k \in \mathbb{N} \cup \{\infty\}$. Suppose that $(r^i + 1) \in R^*$, for each $i \in \mathbb{Z}$. Then the set $M \times \mathbb{Z}_k$, equipped with the operation

$$(a, i) \ast (b, j) = \left(\frac{(r^i + 1)(r^j + 1)}{2 \cdot (r^{i+j} + 1)} \cdot (a + b), i + j\right)$$

is a commutative automorphic loop.

Example

— $M$ a vector space over a field of characteristics different from 2,
— $R = \text{End}(M)$; we see $M$ as an $R$-module
— $r$ an automorphism of $M$,
— $k$ odd.
Loops of odd order

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Small normal subgroup

**Lemma**

*If* \(|K| \leq 3\) *then* \(K \rtimes_{\phi} H\) *is a group.*

**Example**

\(K = \mathbb{Z}_4, H = \mathbb{Z}_2, \phi_{1,1} = 3\)

**Lemma**

*Let* \(K \cong \mathbb{Z}_4\). *Then* \(\phi_{i+j,k} = \phi_{i,k} \circ \phi_{j,k}\).
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Bilinear forms

**Proposition**

Let $K = \mathbb{Z}_{n^2}$, for some $n \in \mathbb{N}$. Let $H$ be an abelian group and let $\alpha : H^2 \rightarrow \mathbb{Z}_n$ be a symmetric bilinear form. We define

$$\varphi_{i,j} : x \mapsto (\alpha(i,j) \cdot n + 1) \cdot x.$$ 

Then $K \rtimes \varphi H$ is a commutative automorphic loop.

**Proposition**

Let $K = \mathbb{Z}_{p^2}$, for some prime $p$. Let $H$ be an elementary abelian $p$-group. Let $\alpha_1, \alpha_2$ be two symmetric bilinear forms $H^2 \rightarrow \mathbb{Z}_p$. Let $Q_1$ and $Q_2$ be two loops obtained from $\alpha_1$ and $\alpha_2$. Then $Q_1 \cong Q_2$ if and only if $\alpha_1$ and $\alpha_2$ are equivalent.
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Classification of bilinear forms

Fact

Let $V$ be a vector space over a finite field $F$ of characteristics $p$. If $p > 2$ then there exist 2 non-degenerate symmetric bilinear forms, up to equivalence, namely

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\begin{pmatrix}
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where $a$ is not a quadratic residue.

If $p = 2$ and $\dim V$ is odd then there exists only one non-degenerate symmetric bilinear form, up to equivalence.

If $p = 2$ and $\dim V$ is even then there exist two such forms, one of them alternating.
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**Bilinear mapping \( \varphi \)**

**Observation**

Let \( \varphi : H^2 \to \text{Aut}(K) \) be bilinear. Then the \( \varphi \) satisfies the conditions of the semidirect product if and only if

1. \( \varphi \) is symmetric,
2. granted,
3. \( \text{Im} \varphi \) is commutative,
4. granted,
5. ???

**Lemma**

Let \( R \) be a unitary ring and let \( n \in \mathbb{N}_0 \). Then the following properties are equivalent:

- there exists \( G \), a commutative subgroup of \( R^* \), such that, for all \( a, b, c \in G \), we have \( na = n \) and \( ab + ac + bc = 1 + 2abc \);
- there exist elements \( x_1, x_2, \ldots \) in \( R \) such that \( nx_i = 0 \) and \( x_i x_j = 0 \), for all \( i, j \).
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Construction with a bilinear mapping

**Theorem**

Let $K$ be an abelian group and let $n \in \mathbb{N}_0$. Let $X$ be a subset of $\text{End}(K)$ satisfying $nX = X^2 = 0$. Denote $G = \langle X + \text{id}_K \rangle_{\text{Aut}(K)}$. Let $\varphi$ be a symmetric bilinear $\mathbb{Z}_n$-module mapping $H^2 \to G$. Then $K \rtimes_\varphi H$ is a commutative automorphic loop.

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$K = \mathbb{Z}_{n^2}, X = \{n\}, G = \{kn + 1; k \in \mathbb{Z}\}.$

**Example**

— $K, H$: vector spaces over a field $F$ of characteristic $n$,
— $M_{ij}$ is a square matrix with 1 on position $i, j$ and 0 elsewhere,
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Bilinear mapping $\varphi$
Loops of order $p^3$

**Proposition**

*There exist at least 6 non-isomorphic commutative automorphic loops of order $p^3$, for $p$ prime, namely*

- $\mathbb{Z}_p^3, \mathbb{Z}_{p^3}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p$,
- $K = \mathbb{Z}_{p^2}, H = \mathbb{Z}_p, X = \{p\}$, $\varphi$ equivalent to the scalar product,
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**Theorem (de Barros, Grishkov, Vojtěchovský)**

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- $K = \mathbb{Z}_2^p$, $H = \mathbb{Z}_p$, $X = \{ \binom{0}{0} \binom{1}{0} \}$, $\varphi$ non-degenerate,
- $K = \mathbb{Z}_2^p$, $H = \mathbb{Z}_2$, $\varphi_{1,1}$ of order 3.

**Theorem (de Barros, Grishkov, Vojtěchovský)**

There exist exactly 7 non-isomorphic commutative automorphic loops of order $p^3$, for $p$ prime.
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