1. Use the Alternating Series Test to check whether the series
\[
\sum_{n=1}^{\infty} (-1)^n \left(e^{\frac{1}{n}} - 1\right)
\] (1)
converges or diverges.

**Solution.** To prove that our series converges, we must show that the sequence

\[ b_n = \left\{ e^{\frac{1}{n}} - 1 \right\} \]

is decreasing, positive, and converges to 0. First note that for every positive integer \( n \), \( b_n \) is positive since

\[
\frac{1}{n} > 0 \implies e^{\frac{1}{n}} > e^0 \quad \text{since } e^x \text{ is increasing}
\]

\[ \implies e^{\frac{1}{n}} > 1 \]

\[ \implies b_n = e^{\frac{1}{n}} - 1 > 0. \]

We can show that \( f(x) = e^{\frac{1}{x}} - 1 \) is decreasing, which will imply that \( b_n \) is decreasing.

\[ f'(x) = -\frac{e^{\frac{1}{x}}}{x^2} < 0 \]

for \( x > 0 \), so \( f(x) \) is decreasing and thus so is \( b_n \). Another way of showing that our sequence is decreasing is the following. For every positive integer \( n \), we have that

\[ n < n + 1 \implies \frac{1}{n} > \frac{1}{n + 1} \]

\[ \implies e^{\frac{1}{n}} > e^{\frac{1}{n + 1}} \quad \text{since } e^x \text{ is increasing} \]

\[ \implies e^{\frac{1}{n}} - 1 > e^{\frac{1}{n + 1}} - 1 \]

\[ \implies b_n > b_{n+1}. \]

Therefore our sequence is decreasing. Furthermore, since

\[ \lim_{n \to \infty} e^{\frac{1}{n}} = 1 \implies \lim_{n \to \infty} b_n = \lim_{n \to \infty} e^{\frac{1}{n}} - 1 = 0, \]

so our sequence converges to 0 as well. Therefore by the Alternating Series Test, the sum (1) converges. \(\Box\)
2. Use the Limit Comparison Test to check whether the series

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + \ln n}$$

converges or diverges.

**Solution.** We know that the sum

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. To see that our sum diverges, notice that

$$\lim_{n \to \infty} \frac{n}{n^2 + \ln n} \cdot \frac{n}{1} = \lim_{n \to \infty} \frac{n^2}{n^2 + \ln n} = \lim_{n \to \infty} \frac{1}{1 + \frac{\ln n}{n}} = 1.$$

We know that this previous limit is 1 because, using L’Hopital’s Rule,

$$\lim_{n \to \infty} \frac{\ln n}{n^2} = \lim_{n \to \infty} \frac{\frac{1}{n}}{2n} = \lim_{n \to \infty} \frac{1}{2n^2} = 0.$$

Therefore by the Limit Comparison Test, our original sum (2) must diverge. \qed