1. For each of the following, either provide an example with the desired properties or explain (via a theorem or fact that we proved in class) why such an example cannot exist. (You DO NOT need to provide a proof that your example has the desired properties!)

(a) A countable closed set

(b) A countable open set

(c) A sequence of numbers in $[0, 1]$ which does not have a subsequence converging to a limit in $[0, 1]$

(d) A sequence of numbers in $\mathbb{R}$ which does not have a subsequence converging to a limit in $\mathbb{R}$

(e) A function which is discontinuous at all $x \in \mathbb{R}$
2. (a) Give the definition of the statement “the set \( S \) is open.”

(b) Prove, using your definition from part (a), that if \( U_1 \) and \( U_2 \) are open sets, then \( U_1 \cap U_2 \) is an open set.
3. (a) Give the definition of the statement “the set $S$ is countable.” (Your answer should involve the existence of a function $f$ with some properties.)

(b) Prove, using your definition from part (a), that the set \{5, 7, 9, \ldots\} of odd integers greater than 3 is countable.
4. (a) Give the definition of the statements “the sequence \((x_n)\) approaches the limit \(x\)” and “the sequence \((x_n)\) is Cauchy.”

(b) Use your definition from part (a) to prove that if \((x_n)\) is a convergent sequence (call its limit \(x\)), then \((x_n)\) is Cauchy.
5. (a) Give any definition of the statement “the function $f(x)$ is continuous at $x = c$.”

(b) Prove, using your definition from part (a), that $f(x) = \sqrt{x}$ is continuous at $x = 9$. 
6. (a) Give the definition of the statement “the set $S$ is compact” which involves sequences (this was the “original” definition of compactness from class.)

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at all $c \in \mathbb{R}$ and $K$ is a compact set, then the set $f(K)$ is compact. You may use any results from class in your proof (other than just quoting the result you’re being asked to prove!)
7. Suppose that $S$ is a nonempty set bounded from above. Describe how to create a sequence $(x_n)$ of elements of $S$ which converges to $\operatorname{sup} S$. You do not have to verify that the sequence you construct converges to $\operatorname{sup} S$, but you do need to actually describe how, for each $n \in \mathbb{N}$, you are defining $x_n$. 